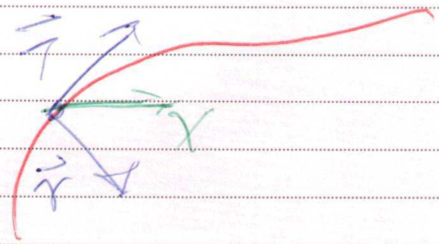
Tangential and Normal Components Acceleration



One can write $a_T$ $a_N$

$$\boxed{\vec{a}(t) = |\vec{r}(t)|' \, \vec{T}(t) + K(t)|\vec{r}(t)|^2 \vec{N}}$$

$$\vec{T} = \frac{\vec{r}'}{|\vec{r}'|} = \frac{\vec{r}}{|\vec{r}|} \Rightarrow \vec{r}(t) = \vec{T}|\vec{r}|$$

$a_T = |\vec{r}|'$ tangential component

$a_N = K \cdot |\vec{r}|^2$ normal component

Chapter 14 - Partial Derivatives

In this Chapter

$f: \mathbb{R}^2 \to \mathbb{R}$ Section 14.1 - Function of 2 variables

Example

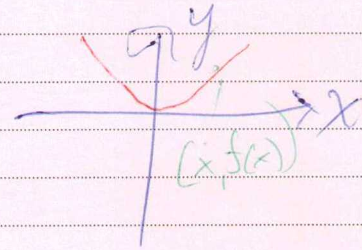
$f(x,y) = x+y$

$f(1,2) = 1+2 = 3$

$f(x,y) = \sin(x) \cdot y + 3$

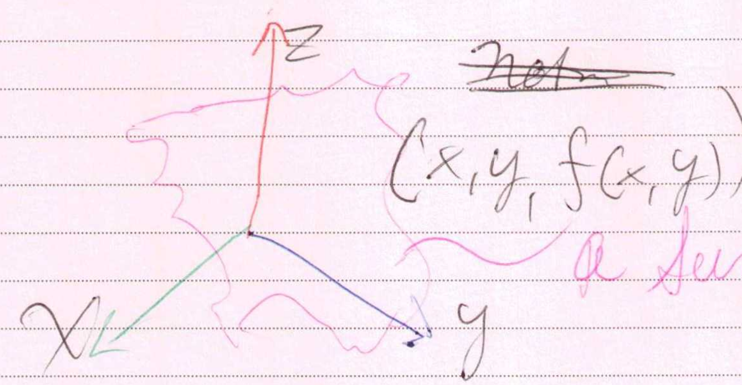
$T(x,y)$ latitude, longitude $=$ temp.

In 1 variable $f(x) = x^3 + x$ , Notation $y = f(x)$




In 2 variables

$$f(x,y) = 2x + y \text{ , Notation } Z = f(x,y)$$




Definition

If $f$ is a function of 2 variables with domain $D$ then the graph of $f$ is the set of all points $(x, y, z)$ in $\mathbb{R}^3$ s.t. $Z = f(x,y)$ and $(x,y)$ is the domain of $f$.

Exercise Let $\boxed{f(x,y)} = 4 - 2x - y$

What type of surface? Its a plane

$\hookrightarrow$ $z = 4 - 2x - y$

$2x + y + z = 4$

Exemple

$$f(x,y)=\sqrt{1-x^2-y^2}$$

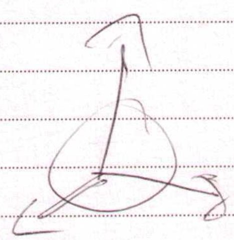
Sketch the graph of $f$

Solutn

$$z=\sqrt{1-x^2-y^2}$$

$$z^2=1-x^2-y^2$$

$$x^2+y^2+z^2=1$$



Domain:
The values $(x,y)$
for which $f(x,y)$ is defined

Exemple

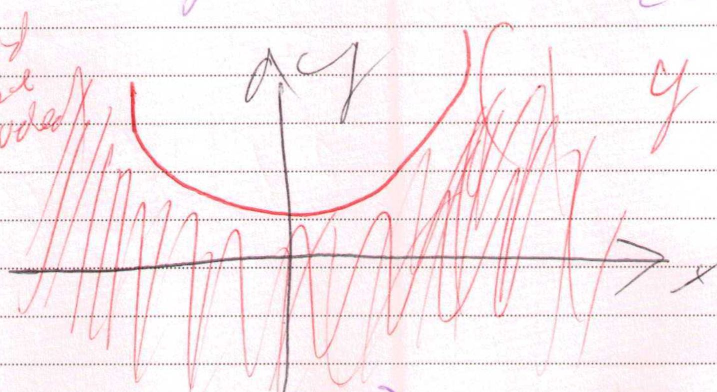
Let $g(x,y)=\sqrt{x^2-y^2+2}$

Domain of $g$

$$x^2-y^2+2\geq 0 \Rightarrow \{(x,y)\mid x^2-y+2\geq 0\}$$

Sketch

Solid line included



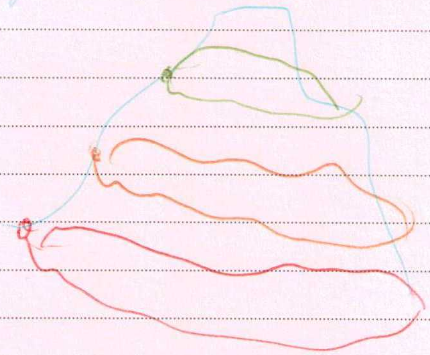

$y=x^2+2$

$y\leq x^2+2$

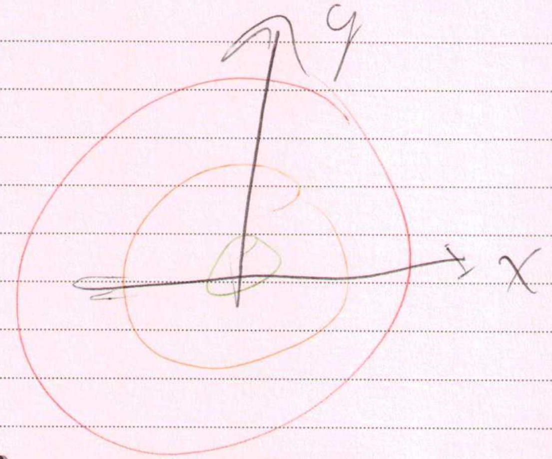
Range of $g$ $[0,\infty)$

## Level Curves

Definition: The level curves of a function f of 2 variables are the curves with equations $f(x,y)=K$ where K is a constant



contour map: a graph of its level curves

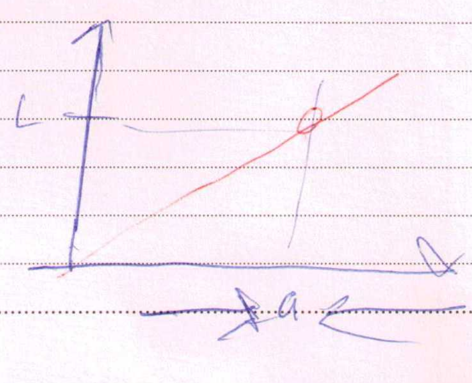



## Section 14.2 - Limits/Continuity

Limits

If $y=f(x)$ we say $\lim_{x\to a} f(x)=L$

and $\exists$ if $\lim_{x\to a^+} f(x)=\lim_{x\to a^-} f(x) \Longrightarrow f(x)=L$

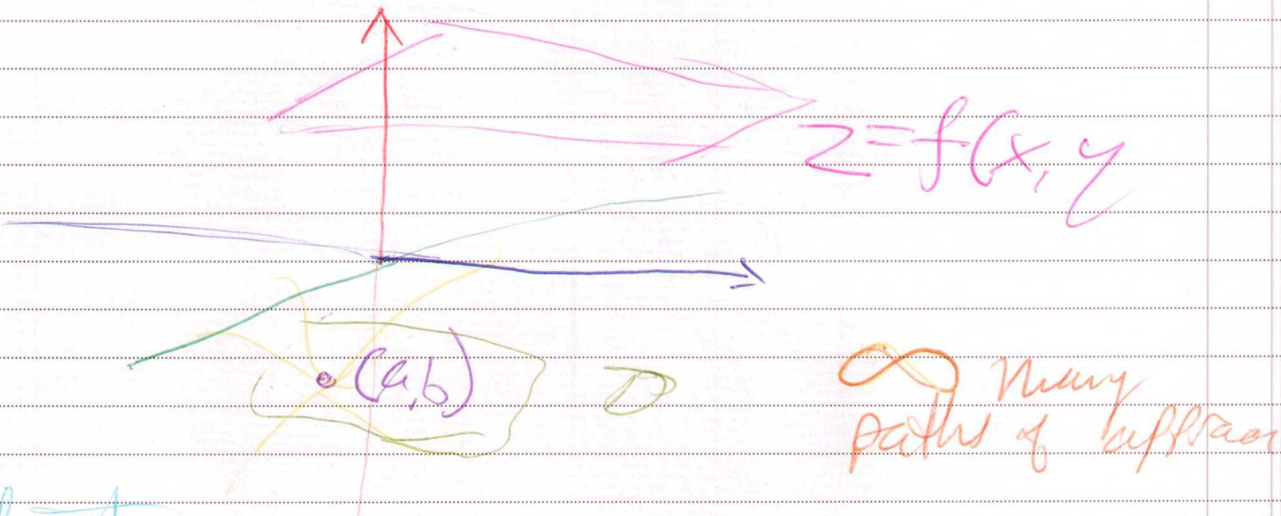



2. paths of approach (Right/Left)

$\mathbb{R}^3$ If $Z = f(x,y)$, if we look at

$$\lim_{(x,y)\to(a,b)} f(x,y) = L.$$

we need limit to be $L$ along $\forall$ path where $(x,y)$ approaches $(a,b)$




Definition
we say $\lim_{(x,y)\to(a,b)} f(x,y) = L$ if:

$$\forall\, \varepsilon > 0 \left\{ \exists\, \delta > 0 \;\middle|\; ((x,y) \in D) \wedge \left(0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta\right) \right\}$$

$$\longrightarrow \left| f(x,y) - L \right| < \varepsilon$$

The value @ $(a,b)$ is irrelevant

---

To show a limit DNE

recall: for a limit to exist, any path of approach to $(a,b)$ has to yield the same limit

$\therefore$ Show 2 paths have unique limits $\rightarrow$

If $f(x,y) \to L_1$ as $(x,y) \to (a,b)$ along path $C_1$ and $f(x,y) \to L_2$ ($L_1 \neq L_2$) as $(x,y) \to (a,b)$ along path $C_2$ Then $\lim_{(x,y)\to(a,b)} f(x,y)$ DNE

Continuity
- $f$ is continuous $(a,b)$ if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$
- $f$ is continuous on $D$ if $f$ is con. at every point in $D$

Fact polynomials are continuous

Exercise

$$\lim_{(x,y)\to(0,0)} xy^2 + 2y - 3x^2 + 4 = 6$$

$$= f(0,1)$$

$$= 0 \cdot 1^2 + 2(1) - 3(0)^2 + 4$$

$$= 2 + 4 = 6 \checkmark$$

Fact $f: \mathbb{R}^2 \to \mathbb{R}$ ^ $g: \mathbb{R} \to \mathbb{R}$ are continuous then $h(x,y) = g(f(x,y))$ is also continuous

Squeeze Thm (aka Sandwich thm)

$$\lim_{(x,y)\to(0,0)} \frac{6x^2y^4}{x^2+y^2}$$

we want to find

$$\ldots \le \frac{6x^2y^4}{x^2+y^2} \le \ldots$$

Consider $y=x$

so $$\lim_{(x,y)\to(0,0)} \frac{6x^6}{2x^2} = 0$$

Since we are told to compute we guess the limit is 0

we know $f(x)$ produces a number that cannot be less than 0

So we bound the bottom by 0

$$0 \le \frac{6x^2y^4}{x^2+y^2} \le \bigcirc$$

notice that $$\frac{x^2}{x^2+y^2} \le \frac{x^2}{x^2} = 1$$

So $$0 \le \frac{6x^2y^4}{x^2+y^2} \le 6y^4 = 1$$

0 OK

By Squeeze thm

$$\lim_{(x,y)\to 0} f(x,y) = 0$$

section 14.3 - partial derivatives

1 variable

$y = f(x)$

2 variable

$z = f(x, y)$

Partial Derivatives : fix one of the variables and to how f changes as the other variable changes

partial derivatives w.r.t $x$

- fix $y = b$ (some constant)
- let $g(x) = f(x, b)$
- Define $\frac{df}{dx} = g'(x)$

partial derivative W.R.T $x$

Example let $f(x) = 3x^2 + 5y + \sin(x)$

- find $\frac{df}{dx}$

$\frac{df}{dx} = 6x + 0 + \cos(x)$

Example let $f(x, y) = 3x^2 y = (3y)x^2$

find $\frac{df}{dx} = (3y) \cdot 2x = 6xy$

constant

Partial Derivative W.R.T $y$

Fix $x = a$ (some constant)

Let $g(y) = f(a, y)$

Define $\frac{df}{dy} = g'(y)$

Example $f(x, y) = 5y + \sin(x)$

$$\frac{df}{dy} = 5 + 0$$

Exercise $f(x, y) = e^{y} - yx^{3}$ constant

Compute $\frac{df}{dy} = e^{y} - x^{3}$

Using limits

$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$\frac{df}{dy} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

notation

$$\frac{df}{dx} = f_x = D_x f$$

$$\frac{df}{dy} = f_y = D_y f$$

Example! $f(x, y) = 4xy + x^2 + y^2$

Compute $f_x(1, 2)$ and $f_y(1, 2)$

Soln: $f_x = 4y + 2x \quad f_x(1, 2) = 4 \cdot 2 + 2 \cdot 1 = 10$

$f_y = 4x + 0 + 2y \rightarrow f_y(1, 2) = 4 \cdot 1 + 2 \cdot 2 = 8$

Higher Order derivatives

$f(x,y) = 4xy + x^2 + y^2$

$f_x = 4y + 2x$

$f_{xx} = 2$ / $f_{xy} = 4$

$f_y = 4x + 2y$ / $f_{yx} = 4$ / $f_{yy} = 2$

$f_{xxx} = 0$

---

Section 14.3

$f(x,y) = 4x^2y^3 + x^3 + 7y$

a) $f_x =$

$f_y =$

10 | 6 | 22

## Theorem ~~Clairaut's Theorem~~

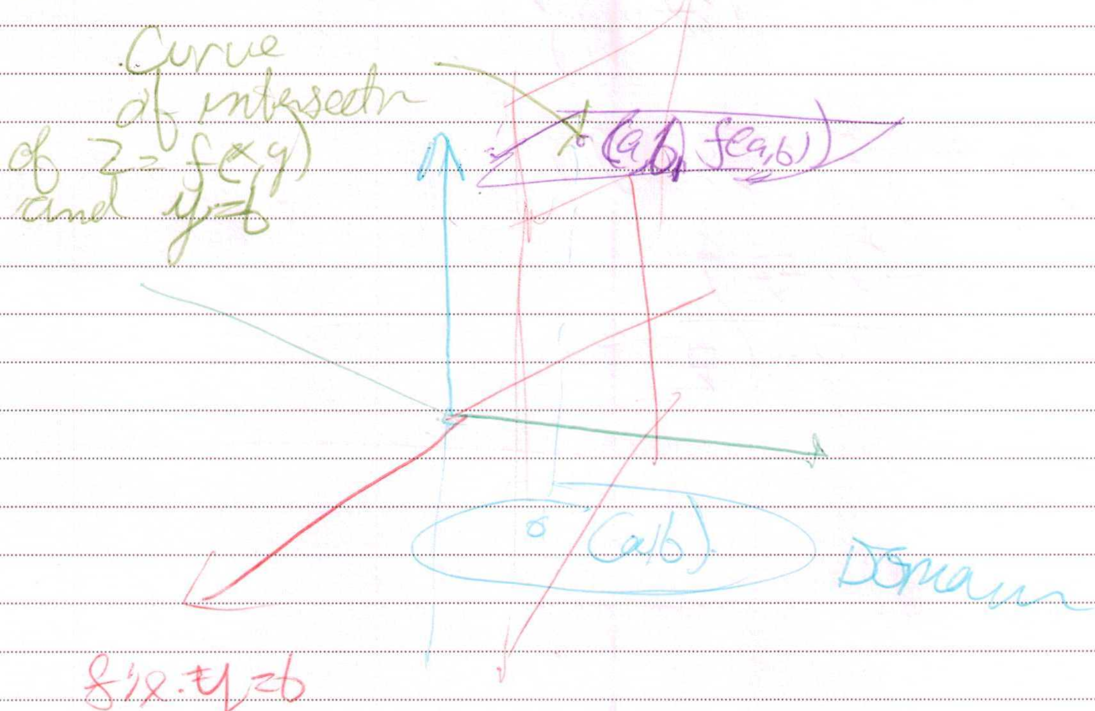
Suppose $f$ is defined on a disk $D$ containing $(a,b)$. If $f_{xy}$ and $f_{yx}$ are both continuous on $D$ then

$$f_{xy}(a,b) = f_{yx}(a,b)$$

## Geometric Interpretation of Partial Derivatives

Consider $f_x(a,b)$

What does this mean in a picture




$f_x(a,b)$ is slope of the tangent line of the curve of intersection @ $(a,b,f(a,b))$

P.P. → Implicit Diff

Find $\frac{dz}{dx}$ $f(x,y)$ → $x^2 + 2y^2 + z \cdot xy = 3z^2$ $3z^2$

We want $\frac{dz}{dy}$ → think y is a constant

think z depends on

Diff w.r.t. x

$$2x + 0 + \frac{dz}{dx} \cdot xy + z \cdot y = 6z \frac{dz}{dx}$$

$(zxy)$ Product Rule

$$\frac{dz}{dx}(xy - 6z) = -2x - 2y$$

$$\frac{dz}{dx} = \frac{-2x - 2y}{xy - 6z}$$

10 6 22

$f_{xx}, f_{xy}, f_{yx}, f_{yy}$

Notation higher derivatives

$f_{xx} = (f_x)_x = \frac{d^2f}{x^2}$

$f_{yy} = (f_y)_y = \frac{d^2f}{dy^2}$

$f_{xy} = (f_x)_y = \frac{d^2f}{dydx}$ } Mixed $2^{nd}$

$f_{yx} = (f_y)_x = \frac{d^2f}{dxdy}$ } Order P.D.

Partial Derivatives of Function 3+ Vars

$f(x, y, z) = 2x + 4yz + 7$

$f_x = 2 + 0 + 0$

$f_y = 0 + 4z + 0$

$f_z = 0 + 4y + 0$

Not implicit Diff b/c one var is not a function of the other var(s)

P. D. Equations

- Diff eqs w/ P.D.s

Laplace's Equation $f_{xx} + f_{yy} = 0$

# Section 14.4 – Tangent Planes and linear approx.

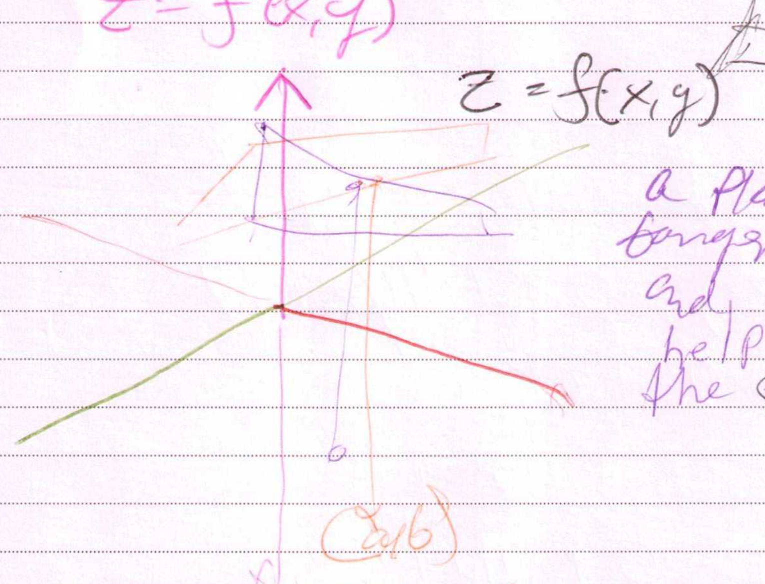
10 | 6 | 22

Idea 

1 var. $y = f(x)$


if you're close enough you can use two points to approx. the function based on the tangent line.

2 vars $z = f(x,y)$




$z = f(x,y)$

a plane / tangent plane and a point help to determine the function in $\mathbb{R}^3$

## Tangent Plane

Let $z = f(x,y)$

Goal: Find the tangent plane of

$z = f(x,y)$ @ $P_0 = (a, b, f(a,b))$

Step 1) take plane $y = b$ to cut the planes $\longrightarrow$ a curve $(C_1)$

Step 2) repeat this step for a second curve w.r.t. the other plane ~~from~~ $y = b$ $(C_2)$

Centroid | 10 | 6 | 22

Look at tangent line to $C_1$ and tangent line to $C_2$ at point $(a, b, f(a,b))$

Tangent plane: plane that contains both tangent ~~lines~~

$C_1$ (intersection) of $z = f(x,y)$ w/ plane $x = a$

$\vec{r}(t) = \langle a, y, f(a,y) \rangle \rightarrow \vec{r}\,'(t) = \langle 0, 1, f_y(a,y) \rangle$

$\vec{T} = \dfrac{\vec{r}\,'(y)}{|\vec{r}\,'(y)|}$

point $(a, b, f(a,b))$

$\vec{r}\,'(b) = \langle 0, 1, f_y(a,b) \rangle$

Step 1 tangent vector line

$L_1 = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ f(a,b) \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ f_y(a,b) \end{pmatrix}$

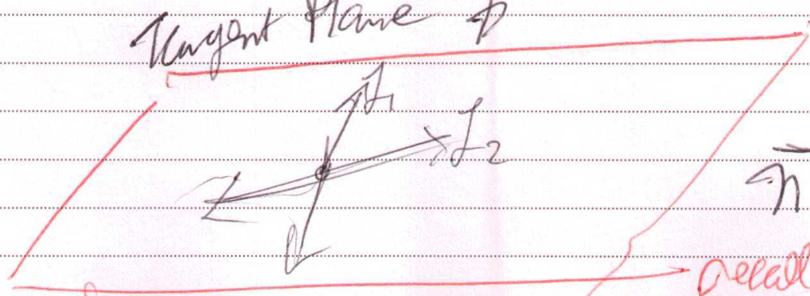
Step 2 curve of intersection plane $y = b$

point $\vec{r}\,'(a) = \langle 1, 0, f_x(a,b) \rangle$

$\vec{r}(x) = \langle x, b, f(x,b) \rangle \rightarrow \vec{r}\,'(x) = \langle 1, 0, f_x(x,b) \rangle$

$L_2 = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ f(a,b) \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ f_x(a,b) \end{pmatrix}$

Tangent Plane $T_P$




point $(a, b, f(a,b))$

$\vec{n} = \vec{v}_1 \times \vec{v}_2$

recall $\vec{v}_1 = \langle 0, 1, f_y(a,b) \rangle$

$\vec{v}_2 = \langle 1, 0, f_x(a,b) \rangle$

$\vec{n} = \langle f_x(a,b), f_y(a,b), -1 \rangle$

$$\begin{pmatrix} f_x(a,b) \\ f_y(a,b) \\ -1 \end{pmatrix} \cdot \begin{pmatrix} x-a \\ y-b \\ z-f(a,b) \end{pmatrix} = 0$$

eq of Tangent plane @ $(a, b, f(a,b))$

$$\boxed{f_x(a,b)(x-a) + f_y(a,b)(y-b) = (z-f(a,b))}$$

$f(x,y) = x^2 - 4y^2$

recall $z = f_x(1,1)(x-1) + f_y(1,1)(y-1) + f(1,1)$

$f_x = 2x \rightarrow f_x(1,1) = 2$

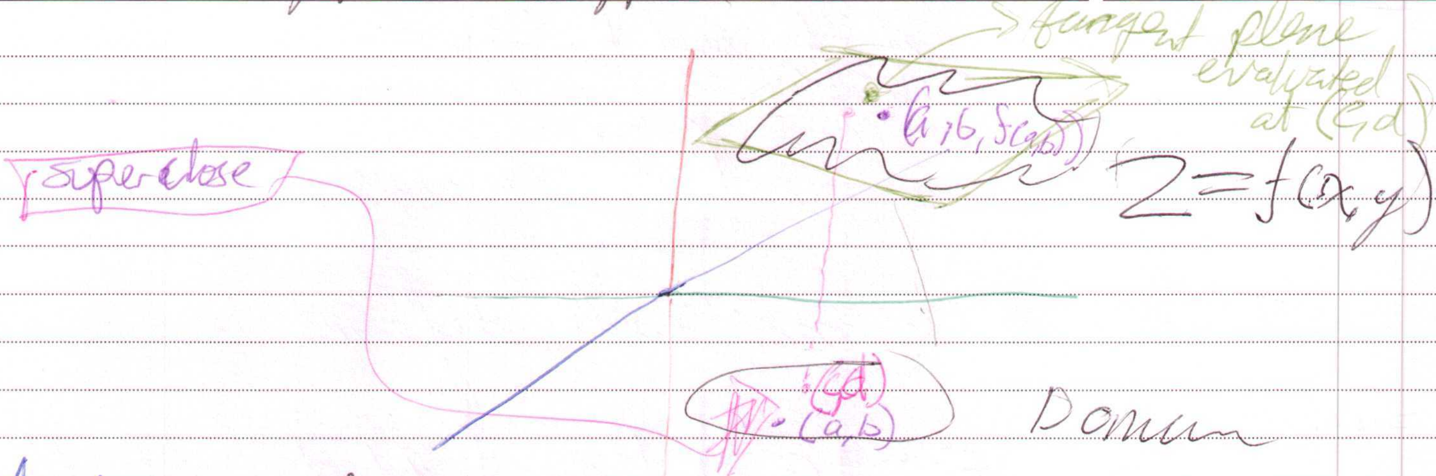
$f_y = -8y \rightarrow f_y(1,1) = -8$

$f(1,1) = 1^2 - 4(1)^2 = -3$

$z = 2(x-1) - 8(y-1) + -3$

# 14.1 Linear approx

10 11 22




$$z = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b)$$

Goal: Approximate the value of $f(c,d)$ using tangent plane to $z=f(x,y)$ @ $(a,b)$

Let $f(x,y) = x^2 - 4y^2$.

Use tangent planes to approximate the value of $f(x,y)$ at $(1.1, 1.1)$

Solution: look at tangent plane at $(1,1)$

Compute tangent plane

$f_x = 2x \quad f_x(1,1) = 2 \qquad f(1,1) = 1-4 = -3$

$f_y = -8y \quad f_y(1,1) = -8$

Tangent plane

$$\boxed{z = 2(x-1) - 8(y-1) + -3}$$

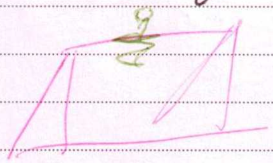
Real value
$f(1.1, 1.1) = 1.1^2 - 4(1.1)^2 = -3.63$

Let's use the plane to approximate $f(1.1, 1.1)$

$f(1.1, 1.1) = 2(1.1-1) - 8(1.1-1) - 3$

$= 0.2 - .8 - 3 = -3.6$

Note: The approximation is good only if $f_x$ and $f_y$ are "nice"



Roof analogy

Tangent plane not well defined

Definition: If $z = f(x,y)$ then $f$ is differentiable at $(a,b)$ if $\Delta z$ can expressed in the form

$$\Delta z = f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where $\varepsilon_1$ and $\varepsilon_2$ are functions of $\Delta x$ and $\Delta y$ such that $\varepsilon_1$ and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$

recall tangent plane

$$\underbrace{z - f(a,b)}_{\Delta z} = f_x(a,b)\underbrace{(x-a)}_{\Delta x} + f_y(a,b)\underbrace{(y-b)}_{\Delta y}$$

Summation: $f$ is differentiable if the linear approximation using tangent ...

Theorem: If $f_x$ and $f_y$ exist near $(a,b)$ and are continuous @ $(a,b)$ then $f$ is differentiable @ $(a,b)$

# 14.5 Chain Rule | 10 | 11 | 22

Case 1: Suppose that $z = f(x,y)$ is a differentiable function of $x$ and $y$ where $\boxed{x = g(t)}$ and $\boxed{y = h(t)}$ and both are differentiable functions of $t$. Then $z$ is a differentiable function of $t$, and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$ Chain Rule

$$z = f(x(t), y(t))$$

Let $z = 2xy + x^2y^3$ where

$x = \sin(t)$ & $y = \cos(t)$

Find $\frac{dz}{dt}$

Solution

$$\frac{\partial f}{\partial x} = f_x = (2y + 2xy^3)$$

$$\frac{dx}{dt} = (\sin(t))' = \cos(t)$$

$$\frac{\partial f}{\partial y} = f_y = 2x + 3x^2y^2$$

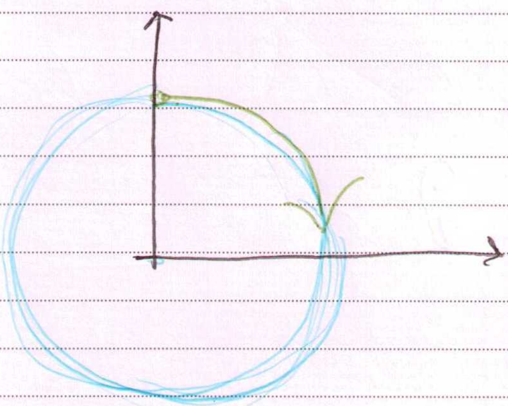
$$\frac{dy}{dt} = (\cos(t))' = -\sin(t)$$

$$\frac{dz}{dt} = (2y+2xy^3)(\cos(t)) + (2x+3x^2y^2)(-\sin(t))$$

# Interpretation

Suppose $z = T(x,y)$ is the temperature at point $(x,y)$

Suppose $x = \sin(t)$ ; $y = \cos(t)$ is time



$\boxed{\text{Case 2:}}$

Suppose $z = f(x,y)$ where $x = g(s,t)$ and $y = h(s,t)$ with $f, g, h$ differentiable then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\wedge \ \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Let $z = e^x + 2xy$

$x = s + 2t \qquad y = st^2$

$\frac{\partial z}{\partial x} = e^x + 2y \quad \Big| \quad \frac{\partial z}{\partial y} = 2x$

$\frac{\partial x}{\partial s} = 1 \quad \Big| \quad \frac{\partial y}{\partial s} = t^2$

$\frac{\partial x}{\partial t} = 2 \quad \Big| \quad \frac{\partial y}{\partial t} = 2st$

$$\frac{dz}{ds} = (e^x + 2y)(1) + (2x)(t^2)$$

$$= e^{s+2t} + 2st^2 + 2(s+2t)t^2$$

$$\frac{\partial z}{\partial t} = (e^x + 2y)2 + 2x + 2st$$

$$= \left(e^{s+2t} + 2(st^2)\right)2 +$$

Case 1

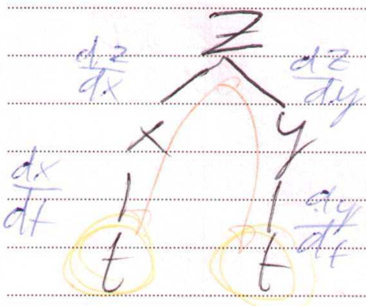
$z = f(x, y)$

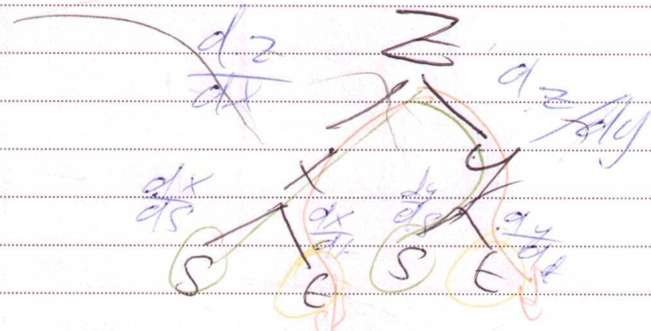
$x(t), y(t)$

Case 2

$z = f(x, y)$

$x(s,t), \; y(s,t)$



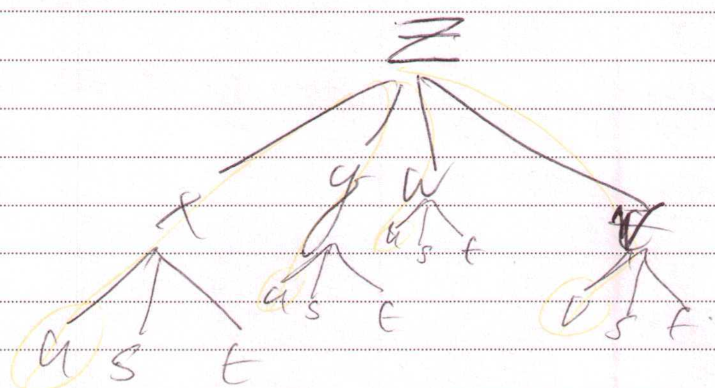

## Chain Rule _ General Case

$z = f(x, y, w, v)$

$x = g(u, s, t) \qquad y = h(u, s, t)$

$w = l(u, s, t) \qquad v = m(u, s, t)$

~~What the~~ what is the formula $\frac{dz}{du}$



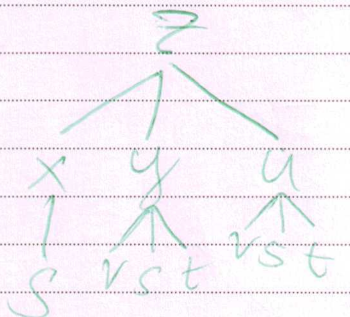

$$\frac{dz}{du} = \frac{\partial z}{\partial x}\frac{dx}{du} + \frac{\partial z}{\partial y}\frac{dy}{du} + \frac{\partial z}{\partial w}\frac{dw}{du} + \frac{\partial z}{\partial v}\frac{dv}{du}$$

$z = x + 2y + 3yu$

$x = 2s \qquad y = 2v + s + t \qquad u = v + s + 2t$

$u = v + s + 2t$

$\frac{dz}{dv}$




$$\frac{dz}{dv} = \frac{dz}{dy}\frac{dy}{dv} + \frac{dz}{du}\frac{du}{dv}$$

$$= 2 \cdot 2 + 3y \cdot 1$$

$$= 4 + 3(2v + s + t)$$

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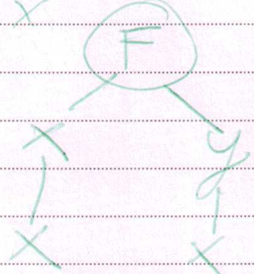
## Section 14.5 - The Chain Rule 10/13/22

Implicit Differentiation: Suppose $F(x,y) = 0$ defines $y$ implicitly as a function of $x$ ($x^3 + y^3 + 6xy = 0$ $f(x,y)$)

Goal: Find $\frac{dy}{dx}$

We'll use the chain rule + write $F(x,y) = 0$ as $F(x, y(x)) = 0$ w.r.t $x$



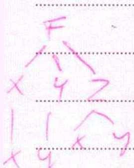

$$\frac{\partial F}{dx} \cdot \frac{dx}{dx} + \frac{\partial F}{dy} \cdot \frac{dy}{dx} = 0$$

$$F_x + F_y \frac{dy}{dx} = 0$$

Solve for $\frac{dy}{dx} \rightarrow \boxed{\frac{dy}{dx} = \frac{-F_x}{F_y}}$

Suppose $F(x, y, z) = 0$ defines $z$ implicitly as func of $x$ & $y$

$x^2 + 2y^3 - z^2 + z = 0$




$$\frac{\partial z}{dx} = \frac{-F_x}{F_z} \qquad \frac{\partial z}{dy} = \frac{-F_y}{F_z}$$

by way of Chain Rule

---

Section 14.6 Directional Derivatives of the Gradient

How is $z = f(x,y)$ changing along some direction of pursuit.

Let $\hat{u} = \langle u_1, u_2 \rangle$ be unit vector,

we want to see how $z = f(x,y)$ changes in the direction of $\hat{u}$

Definition: the direction of derivative of $f$ @ $(x_0, y_0)$ in the direction $\hat{u} = \langle u_1, u_2 \rangle \rightarrow D_u f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + hu_1, y_0 + hu_2) - f(x_0, y_0)}{h}$

$D_u f(x_0, y_0) = \lim_{x \to e} \frac{f(x_0 + hu_1, y_0 + hu_2) - f}{}$

note $D f = f_x \quad \hat{i} = \langle 1, 0 \rangle$

$D f = f_y \quad \hat{j} = \langle 0, 1 \rangle$

$\lim_{h \to 0} \frac{f(x_0 + hu_1, y_0 + hu_2) - f(x_0, y_0)}{h}$

Theorem: If $f$ is a differentiable function of $x$ and $y$, then $f$ has a directional derivative in any direction $\hat{u} = \langle u_1, u_2 \rangle$

and $$\boxed{D_u f(x_0, y_0) = \begin{pmatrix} f_x(x_0, y_0) \\ f_y(x_0, y_0) \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}}$$

- Gradient Vector Definition! $\nabla f(x_0, y_0) = \begin{pmatrix} f_x(x_0, y_0) \\ f_y(x_0, y_0) \end{pmatrix}$

Notice: $D_u f(x, y) = \nabla f(x, y) \cdot \hat{u}$

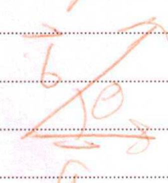
- Maximize the Directional Derivative

In which direction is $D_u f(x_0, y_0)$ largest?

we want $D_u f(x, y) = \nabla f(x, y) \cdot \vec{u}$ to be as big as possible

Recall: $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$ largest when $\cos\theta = 1$

That is!, $\theta = 0$



Directional Derivative is largest when $\vec{u} =$ ______

Theorem: Suppose $f$ is differentiable. The maximum value of $D_u f(x, y)$ is $|\nabla f(x, y)|$ and it occurs when $\hat{u}$ has the same direction as $\nabla f(x, y)$

notice $D_u f(x, y) = \nabla f(x, y) \cdot \vec{u}$

$= |\nabla f(x, y)| \underbrace{|\vec{u}|}_{1} \underbrace{\cos\theta}_{1}$

$= |\nabla f(x, y)|$

# Gradients in 3 Variables

If we have $f(x, y, z)$ then

$$\nabla f(a,b,c) = \begin{pmatrix} f_x(a,b,c) \\ f_y(a,b,c) \\ f_z(a,b,c) \end{pmatrix}$$

Let $f(x,y,z) = xz + 4y$

$$\nabla f(a,b,c) = \begin{pmatrix} z \\ 4 \\ x \end{pmatrix}$$

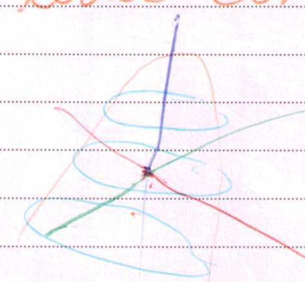
Level curve

$z = f(x,y)$

$f(x,y) = 4$

$f(x,y) = 10$

level curve $z = k$



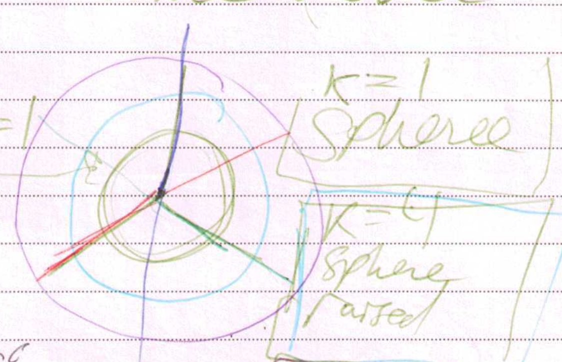
Now $w = f(x,y,z)$

$w = x^2 + y^2 + z^2$ Sketch some level surfaces

Solution $k=1$

$x^2 + y^2 + z^2 = 1$

$k=1$





longitude latitude time

$T = f(x, y, t)$

level surface $f(x,y,t) = 120°$

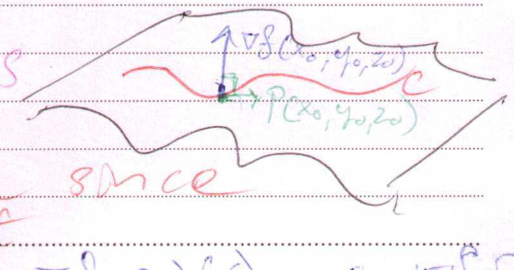
Geometric | let $w = f(x,y,t)$

- The gradient $\nabla f(x_0, y_0, z_0)$ is $\perp$ to the level surface at $(x_0, y_0, z_0)$
- That is, if $P = (x_0, y_0, z_0)$ is a point in the level surface $S: \boxed{f(x,y,z)} = k$ then $\nabla f(x_0, y_0, z_0)$ is $\perp$ to the tangent line to any curve in $S$ through $P$

**Proof** Let $S$ be the surface $f(x,y,z) = k$

Let $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ be curve in $S$ that passes through point $p$, $P = (x_0, y_0, z_0)$

$p = \vec{r}(t_0)$ Then since





$\vec{r} \in S$ $f(x(t), y(t), z(t)) = k$ w.r.t $t$

$$\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = 0 \quad \therefore \quad \nabla f \cdot \vec{r}\,'(t) = 0 \to \nabla f \perp \vec{r}\,'$$

Section 14.6 | 10/18/22

Equation of tangent plane to level surface

$f(x,y,z)=K$ @ $P=(x_0,y_0,z_0)$

$$\begin{pmatrix} f_x(x,y,z) \\ f_y(x_0,y_0, \\ f_z(x_0,y_0,z_0 \end{pmatrix} \circ \begin{pmatrix} x-x_0 \\ y-y_0 \\ z-z_0 \end{pmatrix} = 0$$

$$\vec{n} = \underbrace{\nabla f \ @P}$$

$f_x(x,y,z)(x-x_0) + f_y(x,y,z)(y-y_0) + f_z(x,y,z)(z-z_0) = 0$

Let $w = f(x,y,z) = 2x+y^2+2xz$

suppose $f(x,y,z)=6$

Find tangent plane at $P(1,2,0)$

~~Sol~~

$f_x = 2+2z$ | @ $P(1,2,0)$ $= 2$

$f_y = 2y$ | $= 4$

$f_z = 2x$ | $= 2$

$$\begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix} \circ \begin{pmatrix} x-1 \\ y-2 \\ z-0 \end{pmatrix} = 0$$

$2(x-1)+4(y-2)+2(z-0)=0$

$f(1,2,0) = 2+4+0 = 6$ ✓

---

$S: z = F(x,y)$ Tangent plane @ $(x_0, y_0, F(x_0,y_0))$

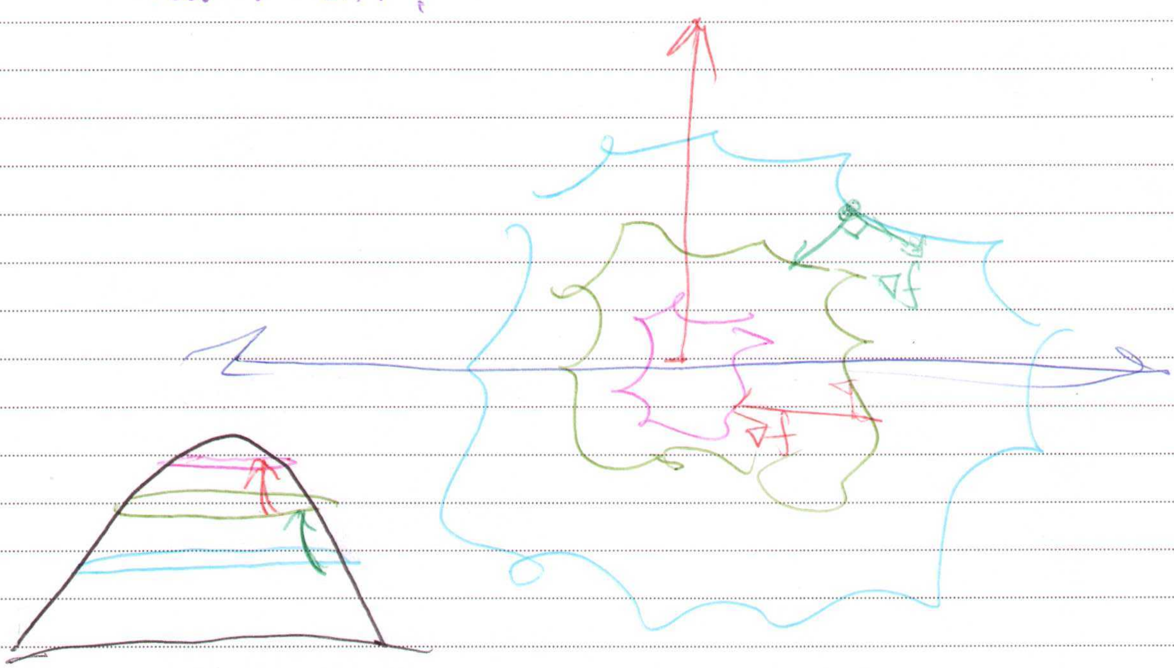
$\therefore z = F_x(x_0,y_0) + F_y(x_0,y_0) + F(x_0,y_0)$

Now $f(x,y,z) = K$

$\longrightarrow F(x,y) - z = 0$

Curve of Steepest Ascent

Fact ! If $z = f(x, y)$ ^ $P = (x_0, y_0, f(x_0, y_0))$ then $\nabla f$ @ P is $\perp$ to the level curve at that Point.



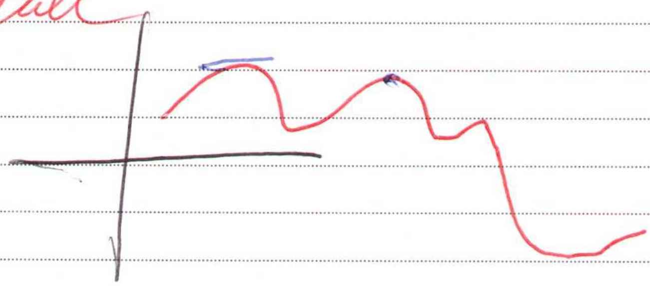

The gradient maximizes the direction of the derivative

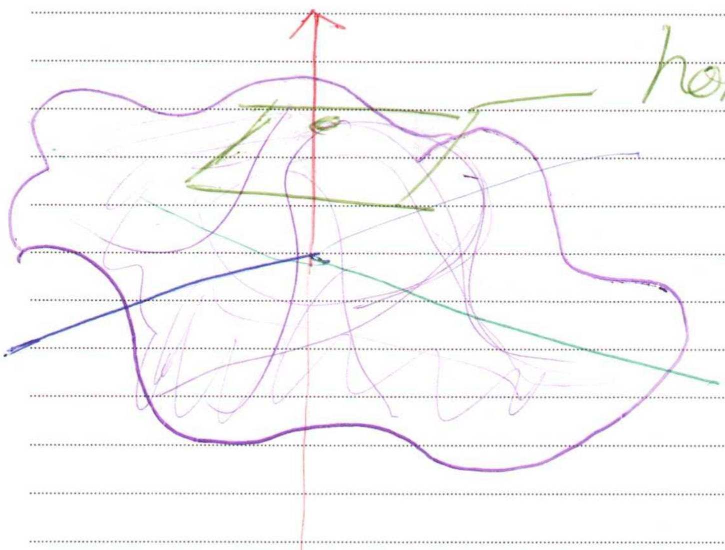
14.7. Maximum / minimum values 10/18/22

Definition
- $f(x, y)$ has a local max @ $(a, b)$ if $f(x, y) \le f(a, b)$ $\forall (x, y)$ close $(a, b)$
- $f(x, y)$ has a local min @ $(a, b)$ if $f(x, y) \ge f(a, b)$ $\forall (x, y)$ near $(a, b)$

Definition
- If $f(x, y) \le f(a, b)$ $\forall (x, y)$ in domain of f, f has an absolute max @ $(a, b)$
- If $f(x, y) \ge f(a, b)$ $\forall (x, y)$ in D $\rightarrow$ than f has an absolute min @ (a,b)

Recall






Theorem If $f$ has a local max or min at $(a,b)$ and the first order partial derivatives of $f$ exist there $\rightarrow$ then $f_x(a,b) = f_y(a,b) = 0$

Tangent plane $(a,b)$ to $z=f(x,y)$

$z = f(a,b)$ constant

Proof Let $g(x) = f(x,b)$

Suppose $f$ has a local max at $(a,b) \rightarrow g$ has a local max at $x=a$

$\therefore g'(a) = 0$

recall $g'(a) = f_x(a,b) = 0$ ← definition of partial derivative

$\therefore f_x(a,b) = 0$ which is what we wanted to show ∎

Definition | 11 | 18 | 22

(a,b) is a critical point of $f(x,y)$ if either $f_x(a,b) = 0$ and $f_y(a,b) = 0$ or if one of the partial derivatives DNE

How do we determine if a critical point is a max or min or neither?

Suppose f has continuous second derivative and suppose $f_x(a,b) = f_y(a,b) = 0$

Let $D(a,b) = f_{xx}(a,b) \cdot f_{yy}(a,b) - [f_{xy}(a,b)]^2$

Then

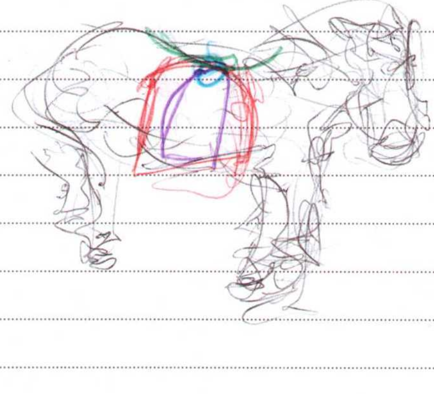
1) If $D > 0$ and $f_{xx}(a,b) > 0$ then $f(a,b)$ is a $\boxed{\text{local min}}$

2) If $D > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is a $\boxed{\text{local max}}$

3) If $D < 0$ then $f(a,b)$ is a saddle point

4) $D = 0$ the test is inconclusive

Saddle Point

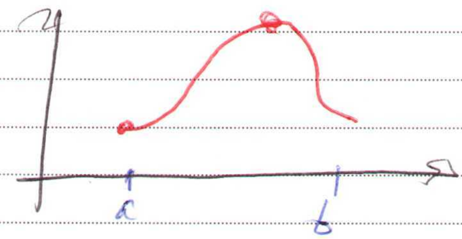


Saddle point is the max in one direction

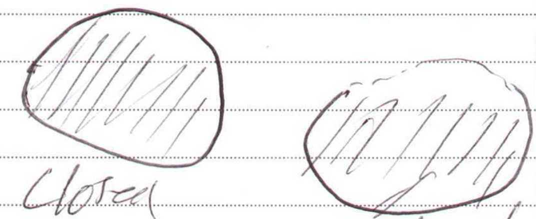
# Absolute Max and Min



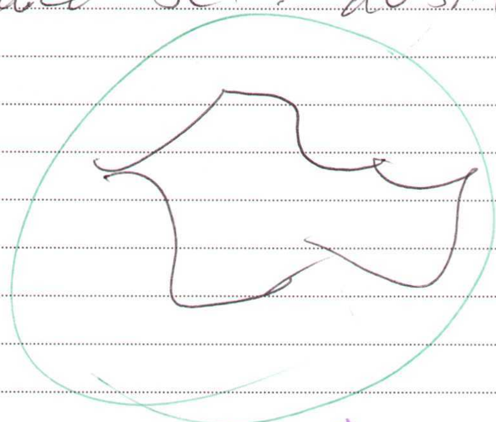
recall




Closed set: a set that contains all its boundary points



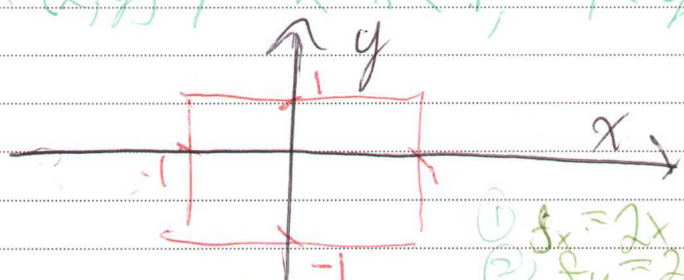

Bounded set: doesn't keep growing



closed bounded set:

**Theorem** If $f$ is continuous on a closed, bounded set $D$ in $\mathbb{R}^2$, then $f$ has an absolute max and an absolute min in $D$. The absolute max / min are either at a critical point or a boundary point (edge)

Find absolute max / min of $f(x,y) = x^2 + y^2 + x^2 y + 4$ where $D = \{(x,y) \mid -1 \le x \le 1,\ -1 \le y \le 1\}$




closed ✓
bounded ✓
continuous ✓

① $f_x = 2x + 2xy = 0$
② $f_y = 2y + x^2 = 0$

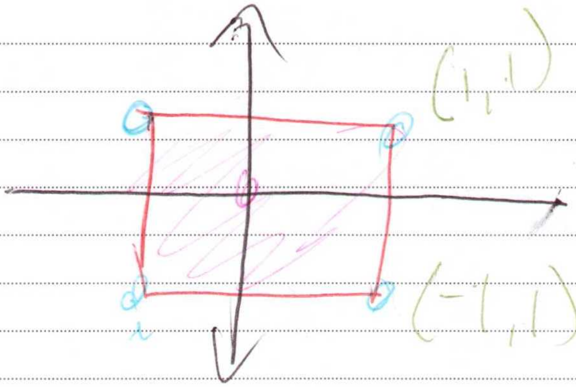
from ① $\Rightarrow 2x(1+y) = 0 \Rightarrow x = 0 \lor y = -1$

from ② $x = 0 \Rightarrow 2y = 0 \Rightarrow y = 0$ $\quad (0,0)$

$y = -1 \Rightarrow -2 + x^2 = 0 \Rightarrow x^2 = 2 \rightarrow x = \pm\sqrt{2}$

$\cancel{(\pm\sqrt{2}, -1)}$ not included

$\boxed{(0,0)}$




(0,0)

$f(x,y) = x^2+y^2+x^2y+4$

$D = \{(x,y) \mid -1 \le$

---

Boundry $x=1$

$f(x,y) = x^2+y^2+x^2y+4$ with end points $(1,-1)$ and $(1,1)$

$f(1,y) = 1^2+y^2+1^2y+4 \rightarrow y^2+y+5$

Let $g(y) = y^2+y+5$

Critical points $g'(y) = 2y+1 = 0 \rightarrow y = -1/2$

Points $\underbrace{(1,-1/2)}_{\text{Critical points}}$ & $\underbrace{(1,-1), (1,1)}_{\text{end points}}$

Boundry $y=1$ w/end points $(1,1), (1,-1)$

Find critical point along $\boxed{y=1}$

$f(x,1) = x^2+1+x^2+4 = 5+2x^2$

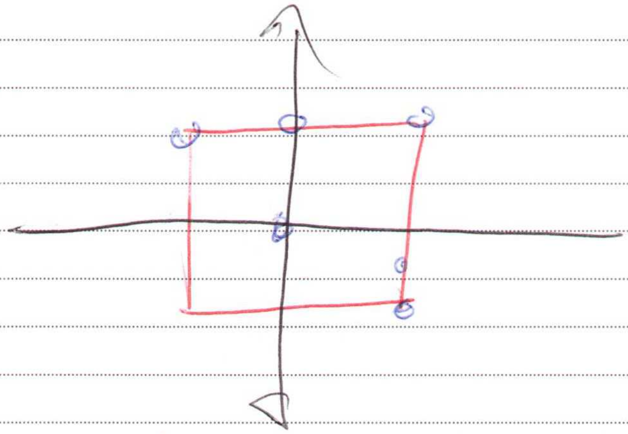
$h(x) = 5+2x^2$

Critical pts $-1 \le x \le 1$

$h'(x) = 4x = 0 \rightarrow x=0$

Point $(0,1)$

10 20 22



Boundry lines

rec: 4
triangle: 3
circle: 1
critical points on end

$f(-1,y) = (-1)^2 + y^2 + (-1)^2 y + 4$

~~$h(x) = 5 + 2x^2$~~ $y^2 + y + 5 = g(y)$

$g'(y) = -\frac{1}{2}$ ~~$h'(x) = 4x \Rightarrow x = 0$~~

~~Point (0,1)~~

Point $(-1, -1/2)$

$f(x,-1) = x^2 + (-1)^2 + x^2(-1) + 4$

$x^2 + 1 - x^2 + 4 = 5$

$g(x) = 5$

$g'(x) = 0$ ! ~~Point (1,0)~~

no critical point

Finally, find absolute max/min

0 - $f(0,0) = 0 + 0 + 0 + 4 = 4$ — smallest

1 - $f(1,1) = 1+1+1+4 = 7$ → largest

2 - $f(-1,1) = 7$

3 - $f(1,-1) = 1+1-1+4 = 5$

4 - $f(-1,-1) = 5$

5 - $f(0,1) = 5$

6 - $f(-1,-1/2) = 1 + 1/4 - 1/2 + 4 = 19/4$

7 - $f(1,-1/2) = 19/4$

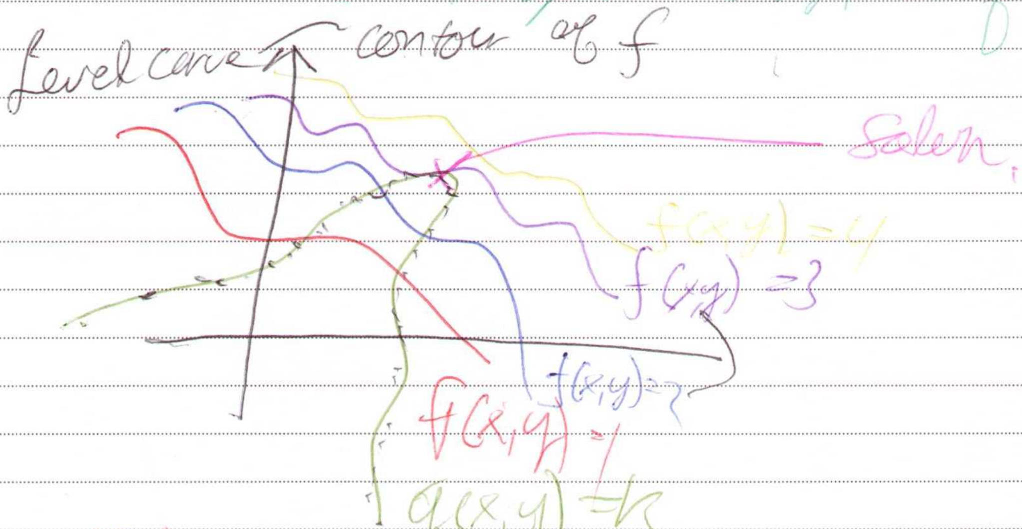
absolute max = 7 @ (1,1) & (-1,1)
absolute min = 4 @ (0,0)

# Section 14.8 - Lagrange Multipliers | 10 | 22

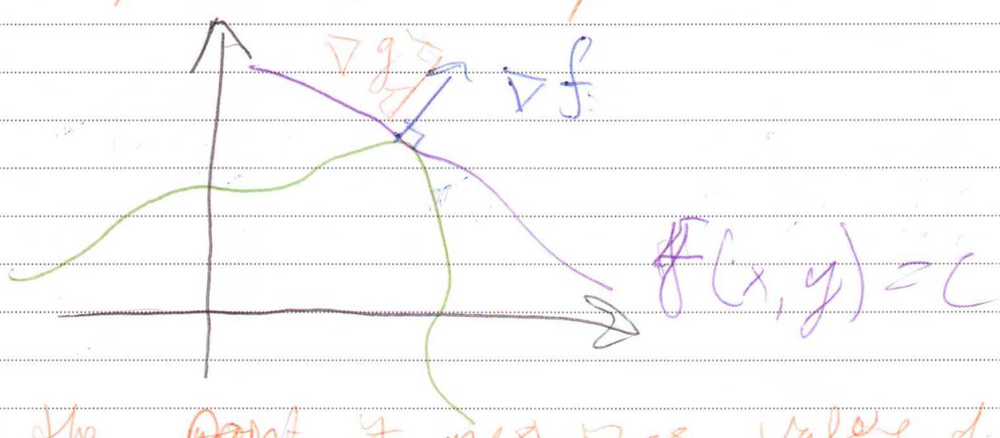
Goal: find the absolute max/min of a function subject to a constraint.

* Find the max/min of $f(x,y)$ subject to the constraints $g(x,y)=k$

$(x,y)$ is restricted to living in the level curve $g(x,y)=k$, and with these points we want the largest $(x,y)$ to $f(x,y)$




Find the point algebraically




At the point it maximizes value of ∇

$\nabla f \parallel \nabla g$ @ $(a,b)$

i.e. $\boxed{\nabla f = \lambda \nabla g}$

Method ~~of~~ of Lagrange multiplier (14.8) | 16 | 20 | 22

Study!!

①. find all values of $x, y$ and $\lambda$ s.t.

$$\& \begin{cases} \nabla f(x,y) = \lambda \nabla g(x,y) \\ g(x,y) = k \end{cases}$$

② Evaluate $f$ at all those points. The largest is the maximum.

Example: Find the max of $f(x,y) = x^2 - xy + y^2$ subject to the constraint $\underbrace{x^2 + y^2}_{g(x,y)} = 2$

$$\nabla f = \begin{pmatrix} 2x - y \\ -x + 2y \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \checkmark$$

$$\nabla g = \begin{pmatrix} 2x \\ 2y \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \checkmark$$

$$\begin{pmatrix} 2x - y \\ -x + 2y \end{pmatrix} = \lambda \begin{pmatrix} 2x \\ 2y \end{pmatrix} \longrightarrow \begin{matrix} 2x - y = \lambda \cdot 2x \\ -x + 2y = \lambda \cdot 2y \end{matrix}$$

$$K = \begin{pmatrix} 1 - y \\ -x + 1 \end{pmatrix}$$

# Solve System of Equations

$$\begin{cases} ① \; 2x - y = \lambda 2x \longrightarrow y = 2x(1-\lambda) \\ ② \; -x + 2y = \lambda 2y \longrightarrow x = 2y(1-\lambda) \\ ③ \; x^2 + y^2 = 2 \end{cases}$$

①/② if $\lambda = 1 \rightarrow y, x = 0$.

③ but $0^2 + 0^2 \neq 2$

$\therefore \lambda \neq 1$

So

Divide ② by $1-\lambda$ $\quad x^2+y^2=2 \Rightarrow$

$$y = \frac{x}{2(1-\lambda)}$$

$$x = 4x(1-\lambda)^2$$

$$x(1-4(1-\lambda)^2) = 0 \quad \therefore x = 0 \text{ or } 1 = 4(1-\lambda)^2$$

If $x = 0$ and $y = \frac{x}{2(1-\lambda)}$

~~then~~ $y = 0$ which contradicts ③ $\therefore x \neq 0$

So $1 = 4(1-\lambda)^2$ is a solution

Case 1

$$1 = 4(1-\lambda)^2$$

$$\tfrac{1}{4} = (1-\lambda)^2 \rightarrow 1-\lambda = \pm \tfrac{1}{2}$$

So $\lambda = \tfrac{1}{2}$ or $\tfrac{3}{2}$

If $\lambda = \tfrac{1}{4}$ $\quad y = \frac{x}{2(1-\lambda)} = \frac{x}{2(1/2)} = x \quad \boxed{y = x}$

Case 2

$x = 3/2$

$$y = \frac{x}{2(1-\lambda)} = \frac{x}{2(1-3/2)} = \frac{x}{2(-1/2)} = -x$$

Eq (3) $x^2+y^2=2$

$x^2+(-x)^2=2$

$2x^2=2$

$x=\pm 1$

$(1,-1)$, $(-1,1)$

Candidates $(1,1)$, $(1,-1)$, $(-1,1)$, $(-1,-1)$

$f(x,y) = x^2 + xy + y^2$

$f(1,1) = 1 - 1 + 1 = 1$

$f(-1,-1) = 1 - 1 + 1 = 1$

$f(1,-1) = \quad = 3$

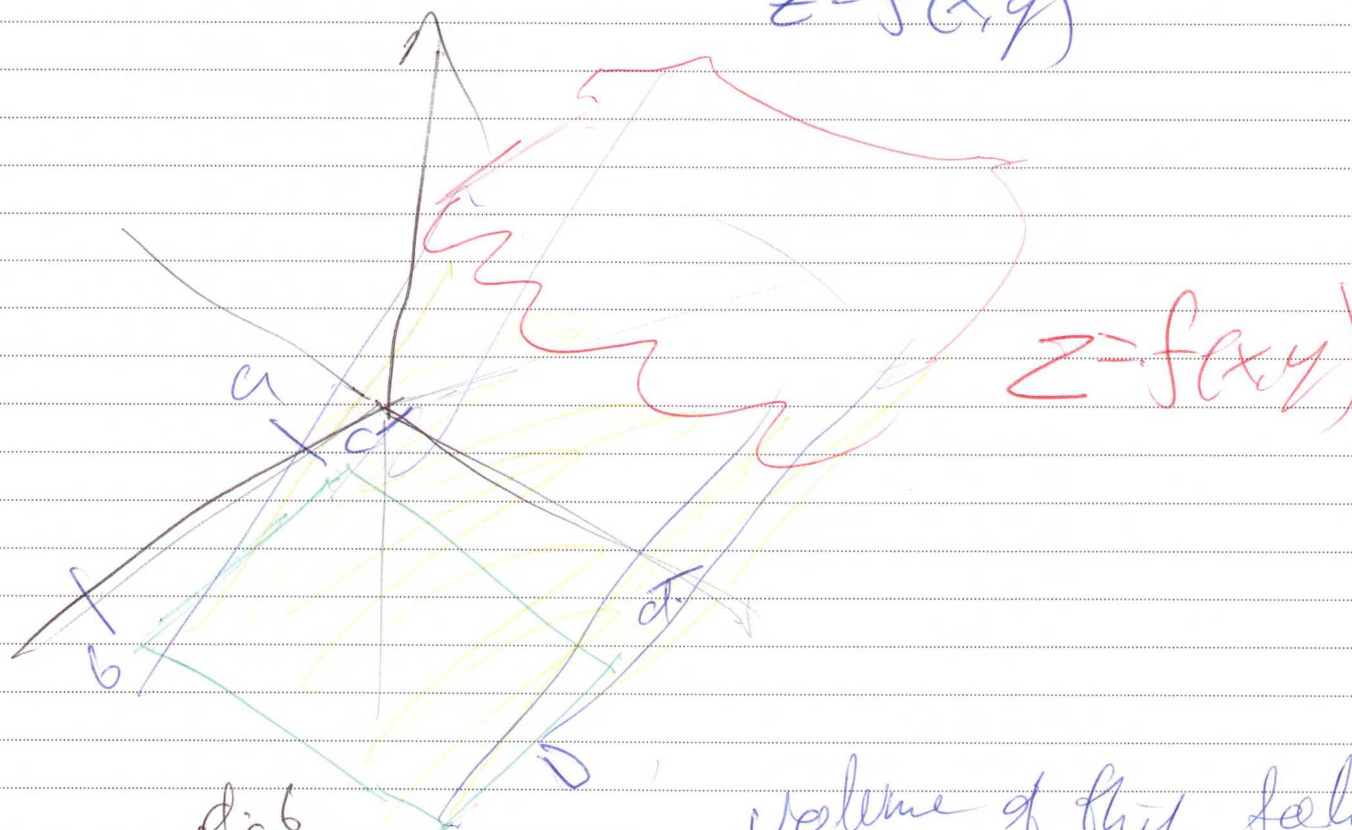
$f(-1,1) = \quad = 3$

$$\begin{cases} g_1(x,y,z)=k_1, \quad g_2(x,y,z)=k_2 \\ \nabla f = \lambda \nabla g_1 + \mu \nabla g_2 \\ g_1(x,y,z)=k_1 \\ g_2(x,y,z)=k_2 \end{cases}$$

The plane $x+y+z=1$ cuts the cylinder $x^2+y^2=1$ in an ellipse. Find the point on the ellipse closest to the origin

$f$ ~ dist to origin

Constraint $\begin{cases} x+y+z=1 \\ x^2+y^2=1 \end{cases}$

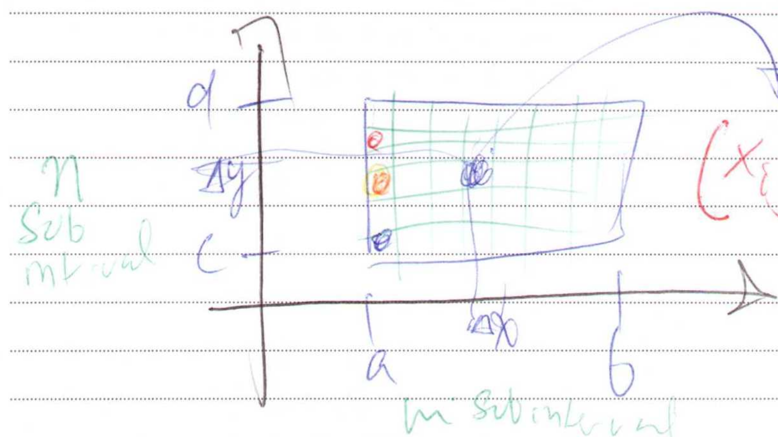



$$\int_c^d \int_a^b f(x,y)\,dx\,dy$$

volume of this solid

Domain $= \{(x,y) \in \mathbb{R}^2 \mid a \le x \le b,\ c \le y \le d\}$

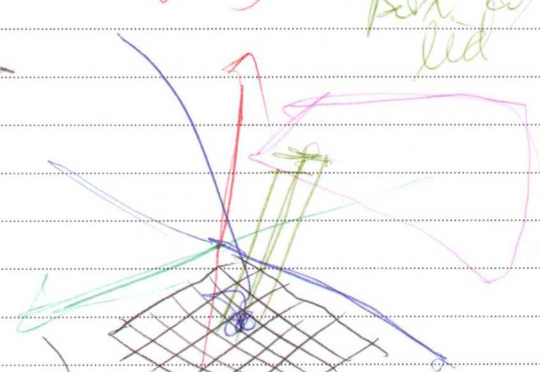
Surface $= \{$




$R_{ij}$ of area $\underline{\Delta x \Delta y}$

$(x_{ij}^*, y_{ij}^*)$ $i \le m$, $j \le n$

Box on flat bed



Volume of green box $\Delta x \Delta y$

Total volume $\approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta x \Delta y$