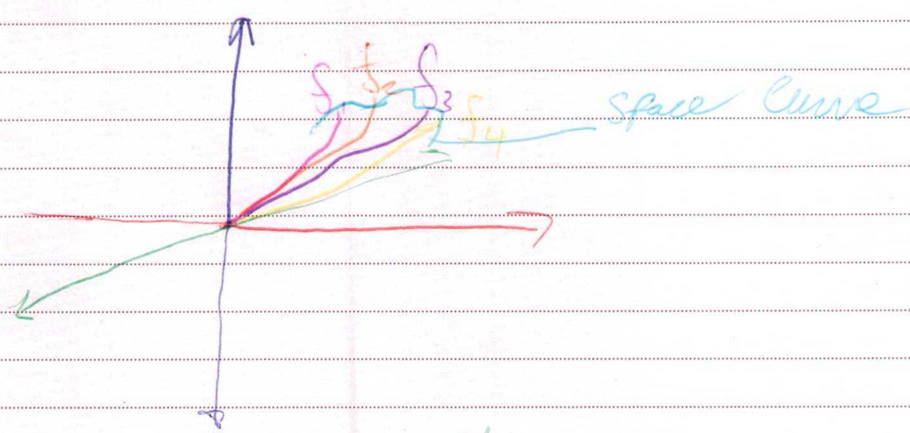
Continuity $\vec{r}$ is continuous at a if

$$\lim_{t \to a} \vec{r}(t) = \vec{r}(a)$$

check that each component function is continuous

Space curves: If you trace the endpoints of the outputs of a function $\vec{r}(t)$ then you get a space which is called a space curve



Example Helix Sketch the curve whose vector equation is $\vec{r}(t) = \begin{pmatrix} \cos(t) \\ \sin(t) \\ t \end{pmatrix}$

$x(t) = \cos(t)$

$y(t) = \sin(t)$

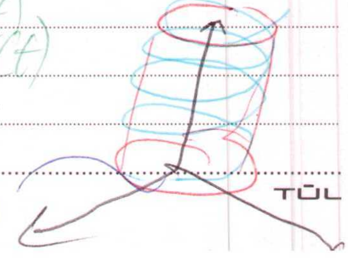
$z(t) = t$

$t = 0.1$

$(\cos(0.1), \sin(0.1), 0.1)$

Notice $x^2 + y^2 = 1$

$t = 0$ $(1, 0, 0)$



Example
Consider $\vec{r}(t) = \langle 1+t, t, 3+2t \rangle$

$$\vec{r}(t) = \langle 1,0,3 \rangle + t\langle 1,1,2 \rangle$$

$$\vec{r}(t) = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

equation of a line

Example
Consider the curve of intersection of surfaces $x^2+y^2=4$ and the plane $4x+3y+6z=12$

$\to z = \frac{12-4x-3y}{6}$

Goal Find $\vec{r}(t)$

$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

$x(t) = 2\cos(t)$
$y(t) = 2\sin(t)$

x^2+y^2
$= 4\cos^2 t + 4\sin^2 t$
$= 4$

$$\vec{r}(t) = \left\langle 2\cos(t), 2\sin(t), \frac{(12-4\cos(t)-3\sin(t))}{6} \right\rangle$$

find the curve of the intersection $x^2+y^2=1$ $z=y^2$

$$\vec{r}(t) = \langle \cos(t), \sin(t), \sin^2(t) \rangle$$

$\vec{r}(t) = \langle 1,1,1 \rangle$

solution
$x(t) = \cos(t)$
$y(t) = \sin(t)$
$z(t) = \sin^2(t)$

Example: Find the curve intersection
$z = x^2 + y^2$ and the plane $z = 2x$

$x^2 + y^2 = 2x$ $\quad x^2 - 2x + y^2 = 0$

$(x-1)^2 - 1 + y^2 = 0$

$(x-1)^2 + y^2 = 1$

$x = \cos(t) + 1$ $\quad x - 1 = \cos(t)$
$y = \sin(t)$

$(x-1)^2 + y^2 = 1$

$z = 2x = 2\cos(t) + 2$

$\vec{r}(t) = \langle \cos(t) + 1, \sin(t), 2\cos(t) + 2 \rangle$

13.2 - Derivates and Integrals of vector functions

If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ where f & g & h are differentiable then $\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle$

Note! $\vec{r}\,'(t)$ is still a vector function and is called the tangent vector to the curve

Exercise: Let $\vec{r}(t) = \langle 3t^3 + 2, \cos(t), e^t \rangle$
Find $\vec{r}\,'(t)$ & $\vec{r}\,''(t)$

$\vec{r}\,'(t) = \langle 9t^2, -\sin(t), e^t \rangle$

$\vec{r}\,''(t) = \langle 18t, -\cos(t), e^t \rangle$

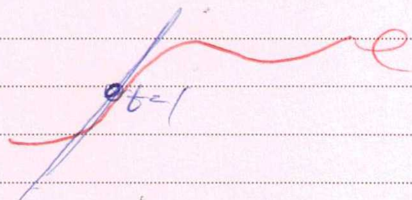
Exercise Let $\vec{r}(t) = \langle 2t, t^2+4t, 5 \rangle$

a) compute $\vec{r}(1)$

$= \langle 2, 5, 5 \rangle$

b) compute $\vec{r}'(1) = \langle 2, 6, 0 \rangle$

c) find the equation of the tangent line to $\vec{r}(t)$ at $t=1$



point 2, 5, 5
dir. vec 2, 6, 0

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 5 \end{pmatrix} + t \begin{pmatrix} 2 \\ 6 \\ 0 \end{pmatrix} \equiv \vec{l}(t) = \langle 2+2t, 5+6t, 5 \rangle$$

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$$

Differentiation Rules

Let $\vec{u}$ & $\vec{v}$ be differentiable vector functions, c a scaler / f is real value function $f: \mathbb{R} \to \mathbb{R}$

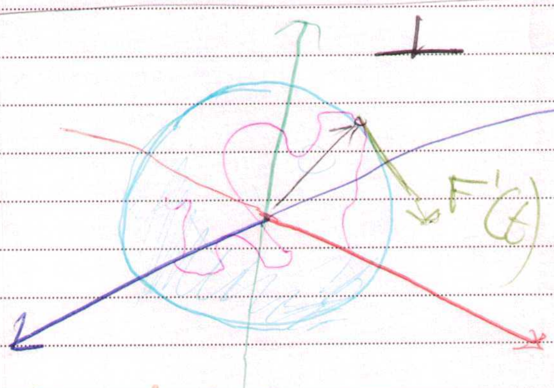
1) $[\vec{u}(t) + \vec{v}(t)]' = \vec{u}\,'(t) + \vec{v}\,'(t)$
2) $[c\vec{u}(t)]' = c\vec{u}\,'(t)$
3) $[f(t)\vec{u}(t)]' = f'(t)\vec{u}(t) + f(t)\,\vec{u}\,'(t)$
4) $[\vec{u}(t) \cdot \vec{v}(t)]' = \vec{u}\,'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}\,'(t)$
5) $[\vec{u}(t) \times \vec{v}(t)]' = \vec{u}\,'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}\,'(t)$
6) $[\vec{u}(f(t))]' = f'(t)\,\vec{u}\,'(f(t))$ ← chain rule

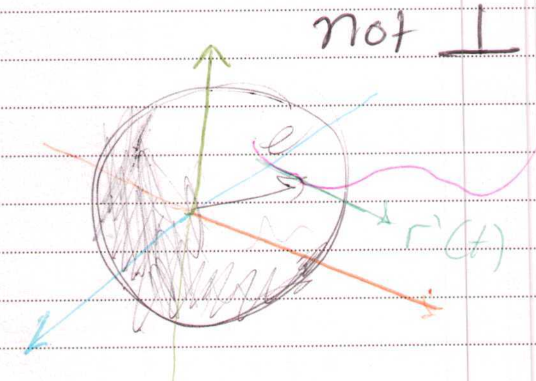
Theorem

$\forall t$ if $|\vec{r}(t)| = c$, $\rightarrow \vec{r}\,'(t) \perp \vec{r}(t)$

$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

$|\vec{r}(t)| = \sqrt{x^2+y^2+z^2} = c$





Proof of Theorem

Goal: Show that if $|\vec{r}(t)| = c \rightarrow \vec{r}\,'(t) \perp \vec{r}(t)$

that is $\vec{r}\,'(t) \cdot \vec{r}(t) = 0$

we know $|\vec{r}(t)| = c$

so $|\vec{r}(t)|^2 = c^2$

$|\vec{r}(t)|^2 = \vec{r}(t) \cdot \vec{r}(t) = c^2$

Differentiate

$$\vec{r}\,'(t) \cdot \vec{r}(t) + \vec{r}(t) \cdot \vec{r}\,'(t) = \underset{(c^2)'}{0}$$

$$2\vec{r}\,'(t) \cdot \vec{r}(t) = 0$$

$$\vec{r}\,'(t) \cdot \vec{r}(t) = 0 \longrightarrow \vec{r}\,'(t) \perp \vec{r}(t) \quad \blacksquare$$

which is what we wanted to prove

Integrals

Let $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

then $\int_a^b \vec{r}(t)\,dt = \left\langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt \right\rangle$

$\left\langle \int 2t+1\,dt, \int \sin(t)\,dt, \int 5e^{2t}\,dt \right\rangle \Rightarrow \left\langle t^2+t \Big|_0^1, -\cos(t)\Big|_0^1, \frac{5e^{2t}}{2}\Big|_0^1 \right\rangle$

If $\vec{R}$ is an antiderivative of $\vec{r}$

that is $\vec{R}'(t) = \vec{r}(t)$

Then

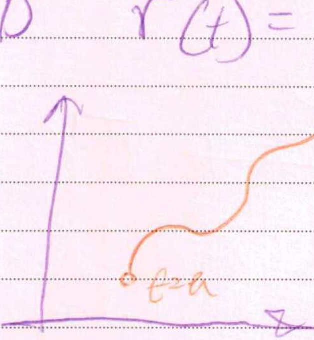
$\int_a^b \vec{r}(t)\,dt = \vec{R}(b) - \vec{R}(a)$

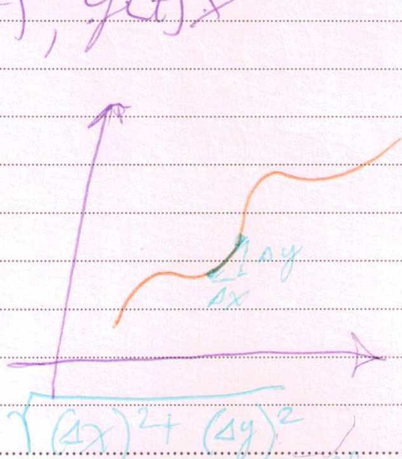
Section 13.3 – Arc Length / Curvature

Arch Length

In 2D $\vec{r}(t) = \langle x(t), y(t) \rangle$

$\int_a^b \sqrt{\left(\frac{dx}{dy}\right)^2 + \left(\frac{dy}{dx}\right)^2}\,dt$





Length of tiny piece $\cong \sqrt{(\Delta x)^2 + (\Delta y)^2}$

Length of whole curve = the sum of all pieces

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle, \quad a \le t \le b$$

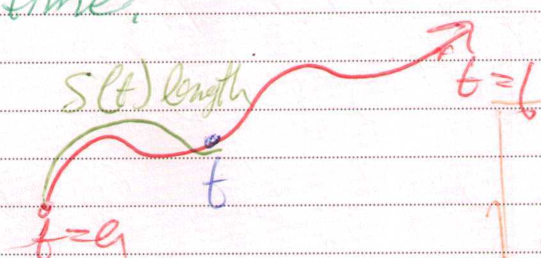
$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

$$L = \int_a^b \left| \vec{r}\,'(t) \right| dt \qquad \left\langle \left(\frac{dx}{dt}\right), \left(\frac{dy}{dt}\right), \left(\frac{dz}{dt}\right) \right\rangle = \vec{r}\,'(t)$$

---

arch length function

What is the length of the curve as a function of time?



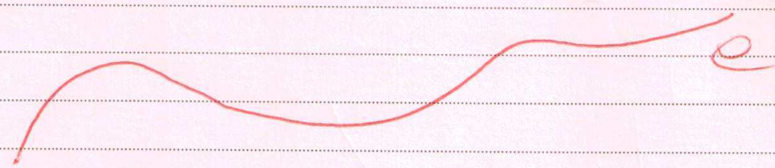
$$\boxed{S(t) = \int_a^t \left| \vec{r}\,'(u) \right| du}$$

Recall $\left(\int_a^t f(u)\, du \right)' = f(t)$

$$\underbrace{\left(\int_a^t \left| \vec{r}\,'(u) \right| du \right)'}_{S'(t)} = \left| \vec{r}\,'(t) \right|$$

Reparametrization of curve

Let $\vec{r}(t) = \langle t, t^2, t^3 \rangle$. $1 \leq t \leq 2$



We can write this curve with a different parameter

$$\vec{r}(v) = \left\langle \frac{v}{2}, \left(\frac{v}{2}\right)^2, \left(\frac{v}{2}\right)^3 \right\rangle$$

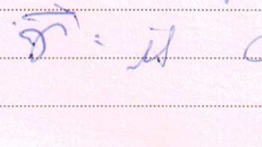
$\vec{r}(2) = \langle 1, 1, 1 \rangle$, $\vec{r}(4) = \langle 2, 4, 8 \rangle$

$\vec{r}(u) = \langle e^u, e^{2u}, e^{3u} \rangle$. $0 \leq u \leq \ln 2$

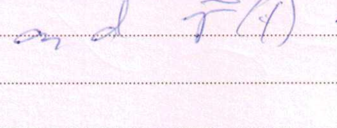
Often one uses less s (arc length) as a parameter, because length arises naturally from the curve.

Definition $\vec{r}(t)$ is smooth on an interval I if $\vec{r}'$ is continuous and $\vec{r}'(t) \neq \vec{0}$ on I

A smooth curve doesn't have a sharp corner or cusps or stop moving



Smooth



not smooth

Reparameterization

Let $\vec{r}(t) = \langle 9t, \cos(t), \sin(t) \rangle$

Compute $s(t)$

$r'(t) = \langle 9, -\sin(t), \cos(t) \rangle$

$$s(t) = \int_0^t \sqrt{9^2 + (-\sin(u))^2 + (\cos(u))^2}\, du$$

$$= \int_0^t \sqrt{81 + 1}\, du$$

$$= \int_0^t \sqrt{82}\, du = \sqrt{82}\, u \Big|_0^t = \sqrt{82}\, t$$

$$s = \sqrt{82}\, t$$

$$t = \frac{s}{\sqrt{82}}$$

$$\vec{r}(t(s)) = \left\langle \frac{9s}{\sqrt{82}}, \cos\left(\frac{s}{\sqrt{82}}\right), \sin\left(\frac{s}{\sqrt{82}}\right) \right\rangle$$

9 27 22

Compute the curvature of a circle w/ radius a,

$r(t) = \langle a\cos(t), a\sin(t) \rangle$

hint $\kappa = \dfrac{|\vec{T}'(t)|}{|r'(t)|}$ recall $\vec{T}(t) = \dfrac{\vec{r}'(t)}{|\vec{r}'(t)|}$

$r'(t) = \langle -a\sin(t), a\cos(t) \rangle$

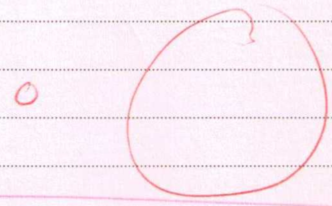
$|r'(t)| = \sqrt{a\sin^2(t) + a\cos^2(t)} = a$

$\vec{T} = \langle -\sin(t), \cos(t) \rangle$

$\vec{T}' = \langle -\cos(t), -\sin(t) \rangle$

$|\vec{T}'| = \sqrt{\cos^2(t) + \sin^2(t)} = 1$

$\boxed{\kappa = \frac{1}{a}}$



Equivilant formulas for curvature

1.) $\kappa = \left|\dfrac{d\vec{T}}{ds}\right|$

2.) $\kappa = \left|\dfrac{\vec{T}'(t)}{r'(t)}\right|$

3.) $\kappa = \dfrac{|r'(t) \times r''(t)|}{|r'(t)|^3}$ ← easier computation

4.) In 2D if $y = f(x)$ then $\kappa(x) = \dfrac{|f''(x)|}{(1 + (f'(x))^2)^{3/2}}$ ← Proof n bolt

$r'(x) = \langle x, f(x) \rangle$

Normal / Binormal vectors

Normal vector

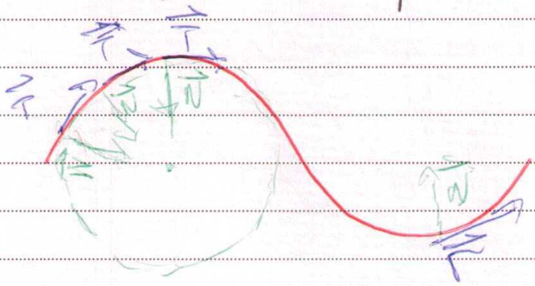
Definition: $\vec{N}(t) = \dfrac{\vec{T}'(t)}{|\vec{T}'(t)|}$ Unit vector in the direction of $\vec{T}'$

Notice $\vec{N} \perp \vec{T}$

Recall if $|\vec{r}(t)| = c \Rightarrow \vec{r} \perp \vec{r}'(t)$

for $|\vec{T}(t)| = 1 \Rightarrow \vec{T} \perp \vec{T}'$

$\therefore \vec{T} \perp \vec{N}$



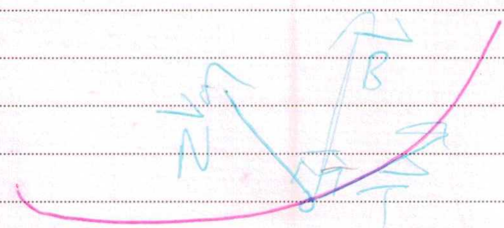
Normal vector points at center of circle

Binormal Vector

we define a 3rd vector $\perp$ to both $\vec{T}$ ^ $\vec{N}$

$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$

"TNB Frame" 3 vectors $\vec{T}, \vec{N}, \vec{B}$



Section 13.4 Motion In Space

Velocity & Acceleration

$\vec{r}(t)$ describes the position of a moving particle

$\vec{v}(t) = \vec{r}'(t)$ (velocity)

$\vec{a}(t) = \vec{r}''(t)$ (acceleration)

Speed $\boxed{|\vec{r}'(t)|}$

Notice that $|\vec{r}'(t)| = |r'(t)| = s'(t)$

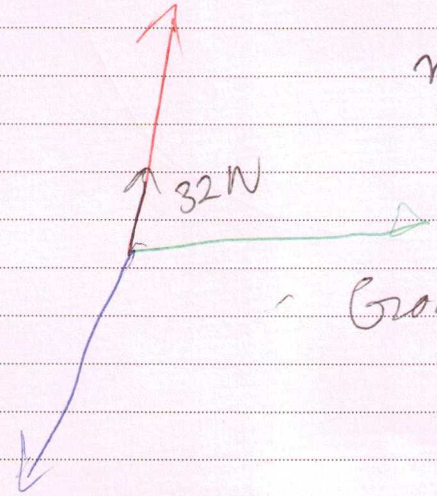
Example

A force with magnitude 32 N acts directly upward from the xy-plane on an object with mass 8 kg

The object starts at the origin with initial velocity $\vec{v}(0) = \langle 1, 1, 0 \rangle$

Find its position function at time t.

Solution



$m = 8kg$
$F = 32$

Goal find $\vec{r}(t)$

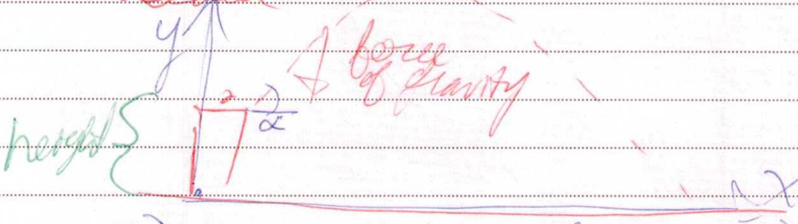
$\vec{r}(0) = \langle 0+D_1, 0+D_2, 0+D_3 \rangle = \langle 0, 0, 0 \rangle$

$D_1 = D_2 = D_3 = 0$

$r(t) = \langle t, t, \frac{4t^2}{2} \rangle$

Projectile Motion

Idea



$\vec{r}'(0)$ initial velocity $\vec{r}'(0)$

$\vec{r}(0)$ initial position $= \langle 0, h \rangle$

α angle of elevation

Goal: Find $\vec{r}(t) = \langle x(t), y(t) \rangle$

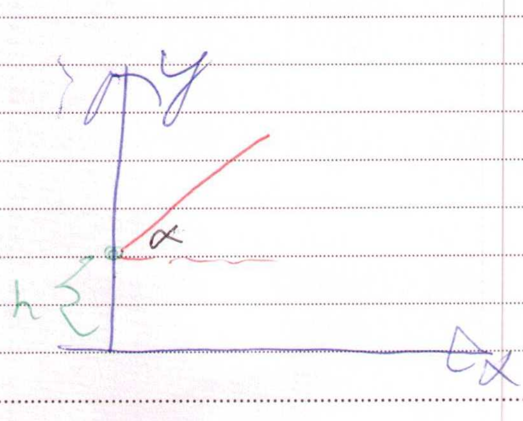
$\vec{r}'' = \langle 0, -g \rangle \quad g \simeq 9.8 m/s^2$

$\vec{F} = m\vec{a} = \langle 0, -mg \rangle$

$\vec{a} = \langle 0, -g \rangle$

$r'(t) = \langle C_1, -gt + C_2 \rangle$

$\cos\alpha = \frac{x}{|\vec{r}_0|} \quad \sin\alpha = \frac{y}{|\vec{r}_0|}$



$x = \cos\alpha |\vec{v}_0| \quad y = \sin\alpha |\vec{v}_0|$

$\vec{r}(t) = \langle |\vec{r}_0| \cos\alpha, -gt + |\vec{r}_0| \sin\alpha \rangle$

$\vec{r}(t) = \langle |\vec{r}_0| \cos\alpha \, t + D_1, \frac{-gt^2}{2} + |\vec{r}_0| \sin\alpha \, t + D_2 \rangle$

Find D_1 + D_2, use $\vec{r}(0) = \langle 0, h \rangle$

$\vec{r}(0) = \langle 0 + D_1, 0 + D_2 \rangle = \langle 0, h \rangle$

$D_1 = 0$, $D_2 = h$

$$\boxed{\vec{r}(t) = \langle |\vec{r}_0| \cdot \cos\alpha \, t, \frac{-gt^2}{2} + |\vec{r}_0| \sin\alpha \, t + h \rangle}$$

→ projectile motion equation

$|\vec{r}_0|$: initial velocity

α : angle of elevation

h : initial height

example

A projectile is fired with initial ~~velocity~~ speed of $\boxed{1000 \text{ m/s}}$ and angle of 45 elevation of 2 meters

a) find the position function of projectile $\vec{r}(t)$

$\vec{r}(t) = \langle 200 \cos(\pi/4) t, \frac{-gt^2}{2} + 200 \sin(\pi/4) t + 2 \rangle$

b) When does projectile hit the ground.

$\frac{-gt^2}{2} + 100\sqrt{2}\,t + 2 = 0$

$t = \frac{\cancel{10} - 100 \pm \sqrt{100^2 - 2(2)}}{1}$

$\vec{r}(t) = \langle 100\sqrt{2}\, t, \frac{-gt^2}{2} + 100\sqrt{2}t + 2 \rangle$

where $g \approx 9.8$