math 340: Elementary Matrices & Linear Alg. | 9 | 8 | 21

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8-10 am MW zoom

(1) Systems of linear equations: — Scaler and constants
(2) matrices: tools
(3) linear transformation:

x = # chickens y = # of Rabbits $x+y=18$

→ Inconsistent : linear Equation has no solution or contradiction $2x+4y=56$

∞ Consistent : linear equation has a solution $x = 18-y$

Homogenous System $2(18-y)+4y=56$

Trivial Solution $36-2y+4y=56$

non-trivial solution: $36+2y=56$

method of elimination: $2(x+y=18)$ $2y=20$ $y=10 \therefore x=8$

$$\begin{cases} 2x+4y=56 \end{cases}$$

$$-2y=(36-56)$$

$$y=10$$

$$\left.\begin{aligned} x+2y+3z&=6\\ 2x-3y+2z&=14\\ 3x+y-z&=-2 \end{aligned}\right\}\ 1$$

$x+2y+3z=6$

$6x-9y+6z=3(14) = 28+14$ (2) $2x+4y+6z=12$

$-6x+2y-2z=-4$ $-2x-3y+2z=14$

$7y+4z=-2$ T

$-11y+8z=42+4$

$-11y+8z=46$ $-11y+8z=46$

$7y+4z=-2$ $-14y+8z=-4$

$-77y+56z=44(7)=308$ $-25y=44$

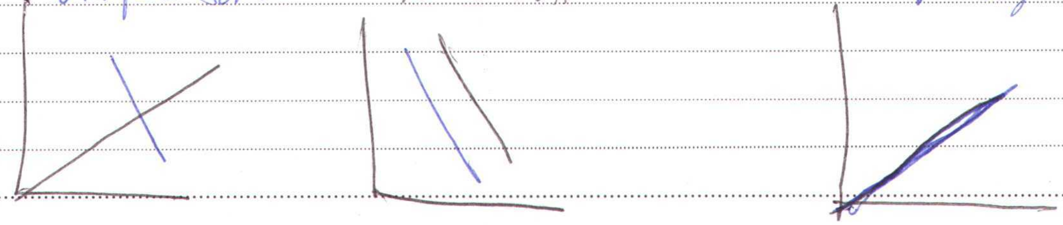
$77y+44z=-22$ $y=-44/25$

$100z=300$

44 × 7 = 308 $\quad$ 44 × 7 = 308

$\boxed{z=3}$ T

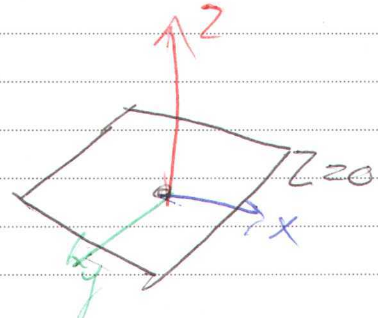
Unique Soln. $\quad$ No Soln. $\quad$ ∞ many Solns



x, y, z $\{z \mid z = 0\}$ xy plane euclidean

$\{x \mid x = 0\}$ yz plane

What are the possibilities for two planes intersecting



a line

a plane

no intersection (parallel)

~~a point~~

↳ needs three constraints

Section 1.1 exercise 1-14.

Example 9/10/2021

vectors

$$\begin{cases} x + 2y + 3z = 6 & (1) \\ 2x - 3y + 2z = 14 & (2) \\ 3x + y - z = -2 & (3) \end{cases}$$

$(2) - 2(1)$

$2x - 3y + 2z = 14$

$-2(x + 2y + 3z) = 6$

$0 - 7 - 4z = 2$

$[2, -3, 2 \mid 14]$

$-2[1, 2, 3 \mid 6]$

$[0, -7, -4 \mid 2]$

Matrices: $\begin{bmatrix} 2 & 6 \\ 4 & 12 \end{bmatrix}$ block of numbers

example

Square Matrix

When matrix has the same number of rows as columns

$A = [a_{ij}]$

i^{th} row j^{th} column

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$\begin{bmatrix} 4 & -2 & 3 \\ 6 & 5 & 2 \end{bmatrix} + \begin{bmatrix} 6 & 2 & -4 \\ 3 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 4+6 & -2+2 & 3+(-4) \\ 6+3 & 5+(-1) & 2+2 \end{bmatrix}$$

You can only add matrices of same dimensions

→ Convention suggests 1×1 matrix by columns

$3 \times \begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix} = \begin{bmatrix} 3(m_1) & \cdots \end{bmatrix}$

TÜL

~~Colunms Dead~~

Linear Combinations!

$$c_1A_1 + c_2A_2 \ldots\ldots + c_nA_n$$

Exercise: let

$$A = \begin{bmatrix} 4 & -2 & 3 \\ 6 & 5 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 6 & 2 & -4 \\ 3 & -1 & 2 \end{bmatrix}$$

Compute $3A + 2B$

$$\cancel{\begin{bmatrix} 3(4)+2(6) & 3(-2)+2(2) & 3(3)+2(-4) \\ 3(6)+2(\ ) & 3(5)+2(-1) & 3(2)+2(2) \end{bmatrix}}$$

$$\begin{bmatrix} 12 & -6 & 9 \\ 0 & 15 & 6 \end{bmatrix} + \begin{bmatrix} 12 & 4 & -8 \\ 6 & -2 & 4 \end{bmatrix} = \begin{bmatrix} 24 & -2 & 1 \\ 6 & 13 & 10 \end{bmatrix}$$

Matrix multiplication:

$$\underset{m \times n}{A} \quad \underset{p \times q}{B}$$

need $n = p$

result $m \times q$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ -1 & 5 \end{bmatrix} \quad B = \begin{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix} & \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix} \end{bmatrix}$$

3×2 $\quad$ 2×1 $\vec{v}_1$ $\quad$ C

$A\vec{v}_1$ and AC

$$A\vec{v}_1 = \begin{bmatrix} 1 \times (-2) + 2 \times (3) \\ 3 \times (-2) + 4 \times (3) \\ -1 \times (-2) + 5(3) \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 17 \end{bmatrix}$$

$$\begin{bmatrix} 1\times3 + 2\times2 & 1\times4 + 2\times1 \\ 3\times3 \quad 4\times2 & 3\times4 \quad 4\times1 \\ -1\times3 \quad 5\times2 & -1\times4 \quad 5\times1 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 17 & 16 \\ 7 & 1 \end{bmatrix}$$

dot product of A & B

$$\begin{bmatrix} 4 & 7 & 6 \\ 6 & 17 & 16 \\ 17 & 7 & 1 \end{bmatrix}$$

note

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

coefficient matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -3 & 2 \\ 3 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \\ 2 \end{bmatrix} = \vec{b}$$

matrix

$$= \begin{bmatrix} x + 2y + 3z \\ 2x - 3y + 3z \\ 3x + y - 1z \end{bmatrix} = x \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + y \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + z \cdot \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

vector

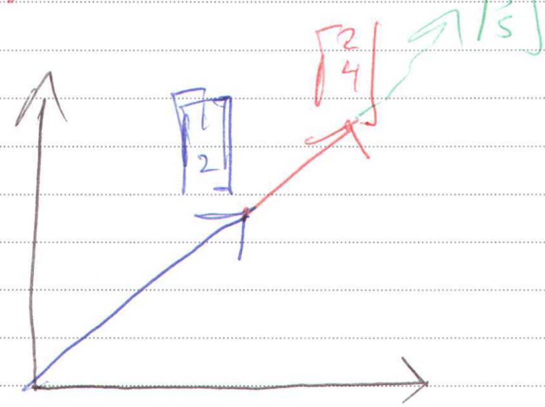
Augmented Matrix

$$\left[A \;\middle|\; \begin{matrix} 6 \\ 14 \\ -2 \end{matrix} \right]$$

A system of linear equations is consistent if and only if the $\vec{b}$ is a linear combination of the column vectors of the coefficient matrix (e.g., A).

$$\begin{cases} x + 2y = 3 \\ 2x + 4y = 3 \end{cases}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$



9 13 21

Matrices are just like numbers except $AB \neq BA$
(cancelation rules don't apply) not commutative

Matrix addition: $A+B = B+A$
$(A+B)+C = A+(B+C)$

$m \times n$

$A + 0 =$

proof:

$A = [a_{ij}]$ $B = [b_{ij}]$

$\forall_{ij}$ $[A+B]_{ij} = a_{ij} + b_{ij}$

$[B+A]_{ij} = b_{ij} + a_{ij}$

$\forall_{ij} [A+B]_{ij} = [B+A]_{ij}$

$\rightarrow A+B = B+A$

exemple
$A: m \times n$
$B: p \times q$
AB iff $m \to p$
$n \to q$

use this to prove

$(rs)A = r(sA)$
$(r+s)A = rA + sA$
$r(A+B) = rA + rB$
$A(rB) = rAB = (rA)B$

- matrix multiplication remains associative
- matrix multiplication is distributive on both sides

$(A+B)C = AC + BC$

$[AB]_{ij} = [a_{i1} \ldots a_{in}] \begin{bmatrix} b_{1j} \\ \vdots \\ b_{nj} \end{bmatrix} = a_{i1}b_{1j} + a_{i2}b_{2j} \ldots$

$= \sum_{k=1}^{n} a_{ik}b_{kj}$

$A = m \times n$
$B = n \times p$

Sometimes summations look different but define the same sum eg.

$A = \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ a_{m1} & \ldots & a_{mn} \end{bmatrix}$

$\sum_i a$

$\sum_j \sum_i a_{ij}$

$\sum_j \sum_i a_{ji}$

$\sum_{k=1}^{n} a_{ik} \cdot b_{kj}$

Rem: if $A^2 = 0 \not\Rightarrow A = 0$

Distributive

$$A(B+C) = AB + AC$$

$$[A(B+C)]_{ij} = \sum_k a_{ik} [B+C]_{kj}$$
$$= \sum_k a_{ik}(b_{kj} + c_{kj})$$
$$= \sum_k a_{ik} b_{kj} + a_{ik} c_{kj}$$
$$= \sum_k a_{ik} b_{kj} + \sum a_{ik} c_{kj}$$
$$= [AB]_{ij} + [AC]_{ij} = [AB + AC]_{ij}$$

Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 0\times0 + 0\times0 & 0\times1 + 1\times0 \\ 0\times0 + 0\times0 & 0\times1 + 0\times0 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Associativity $\rightarrow (AB)C = A(BC)$

$$[(AB)C]_{ij} = \sum_k [AB]_{ik} \cdot c_{kj}$$
$$= \sum_k \left(\sum_l a_{il} b_{lk}\right) \cdot c_{kj}$$
$$= \sum_k \sum_l a_{il} b_{lk} c_{kj} = \sum_k a_{ik} [BC]_{kj}$$
$$= \sum_k a_{ik} \left(\sum_l b_{kl} c_{lj}\right)$$
$$= \sum_k \sum_l a_{ik} b_{kl} c_{lj}$$

~~Let $l \sim kj$~~

Matrix Transpose: $(A^T)^T = A$; $\underset{m\times n}{A} = [a_{ij}] \rightarrow \underset{n\times m}{A^T} = [a_{ji}]$

$\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$\vec{x}^T = [1\ 2\ 3]$

① Commutes w/ addition
$(A+B)^T = A^T + B^T$

② commutes with scalar
$(rA)^T = rA^T$

③ anti-commutes with multiplication
$(AB)^T = B^T A^T$

order matters

$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

flip

A $m\times n$	A^T $n\times m$
B $n\times p$	B^T $p\times n$
AB $m\times p$	$B^T A^T$
$(AB)^T$ $p\times m$	$p\times m$

Math 340 Diagonal Matrix | 9 | 15 | 21

Diagonal | Scalar | Identity

$$rI_n = \begin{bmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \end{bmatrix} \quad \begin{bmatrix} 3 & 1 & 2 \\ 0 & 5 & 0 \\ 0 & 3 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

upper triangle

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -7 & -4 \\ 0 & 0 & 6 \end{bmatrix}$$

0's upper triangle

soln. set as above

$$\begin{cases} x + 2y + 3z = 6 \\ 2x - 3y + 2z = 14 \\ 3x + y - z = -2 \end{cases} \quad \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -2 & -2 \end{array}\right]$$

$$x + 2y + 3z = 6$$
$$-7y - 4z = 2$$
$$-5y - 10z = -20$$

Symmetric / Skew-symmetric

☆ $A^T = A$ Symmetric $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$

Skew-symmetric iff $a_{ij} = -a_{ji}$

☆ $A = -A^T$ $\begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix}$

Fact: any square matrix A can be written uniquely as a sum $B + C$ s.t. B sym, C skew-sym.

$B = \frac{1}{2}(A + A^T)$ & $C = \frac{1}{2}(A - A^T) \longrightarrow B + C = A$

$\Rightarrow$ check $B = B^T$ $\quad B^T = \left[\frac{1}{2}(A + A^T)\right]^T = \frac{1}{2}\left[A + A^T\right]^T = \frac{1}{2}\left(A^T + (A^T)^T\right)$

$$= \frac{1}{2}(A^T + A)$$
$$= B$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 1\cdot 2+0\cdot 5 & 3\cdot 0+1\cdot 7 \\ 5\cdot 1+5\cdot 0 & 7\cdot 0+7\cdot 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$$

In $\mathbb{R}^n$ we use $\vec{e}_i$ to denote the length n vectors with 1 at the i^{th} entry and 0 elsewhere

e.g. $\mathbb{R}^4$ $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ $\vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ $\vec{e}_i \cdot \vec{x} = x_i$

Effect of Dot product

i^{th} entry of $\vec{x}$

A= $m \times n$ matrix

$$\begin{bmatrix} \vdots & & \vdots \\ \vec{v}_1 & \cdots & \vec{v}_n \\ \vdots & & \vdots \end{bmatrix} \qquad I_m = \begin{bmatrix} \cdots & \vec{e}_1^{\,T} & \cdots \\ & \vdots & \\ \cdots & \vec{e}_m & \cdots \end{bmatrix}\begin{bmatrix} & \vdots & \\ \cdots \vec{e}_i & \vec{v}_j & \\ & \vdots & \end{bmatrix}$$

i^{th} row

j^{th} col

$$\vec{e}_i^{\,T} \vec{v}_j = \vec{e}_i \cdot \vec{v}_j \qquad a_{ij}$$

$$[I_m A]_{ij} = a_{ij} \longrightarrow I_m A = A$$

invertable identity $\rightarrow$

For $n \times n$ "Square Matrices" B is the inverse of A ($B = A^{-1}$) if $AB = I_n$

Fact: if $AB = I_n \leftrightarrow BA = I_n$

if no soln. or ∞ soln. then A is not invertable.

$$A^{-1}(A\vec{x}) = I_n\vec{x} = \vec{x} = A^{-1}b$$

inverse / pre-image

Math 340 | 9/07/21

$$A \cup B := \{x \mid x \in A \lor x \in B\}$$

$$A \cap B := \{x \mid x \in A \land x \in B\}$$

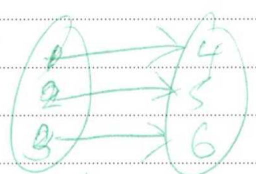
$f: A \to B$ — A is the domain, B is the co-domain

The subset of B "hit" by the function is the range or the image of f

$$im(f) := \{y \in B \mid f(x) = y \text{ for } x \in A\}$$

each element is "hit" — Surjective

each input has an output — Injective

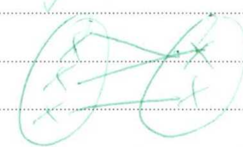


Injective
Surjective

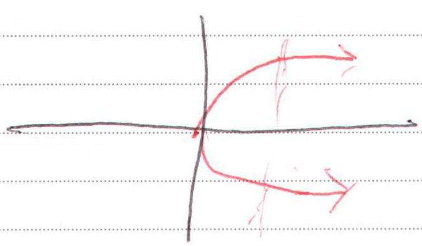
$z = \{5, 6\} \subseteq B$

$f^{-1}(z) = \{1, 2, 3\}$

Injective

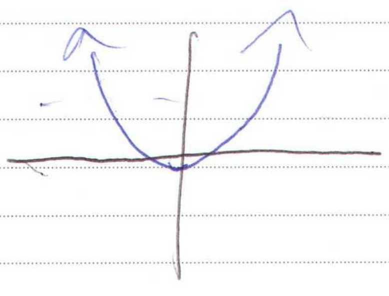


Surjective



not a function

x has more than an output



is a function

$f \circ g$

$f(g(x))$

if $f: A \to B$ then $g: B \to A$
if $f \circ g = id_B$ then $g \circ f = id_A$

not a function $\longrightarrow$ $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x^2 \\ x + y \\ \sin(z) \end{bmatrix}$

A function only has an inverse when both injective and surjective

Define a function $\boxed{f: \mathbb{R}^2 \to \mathbb{R}^2}$ by asking $f(\vec{x})$ to be orthogonal projection to the line $y=x$

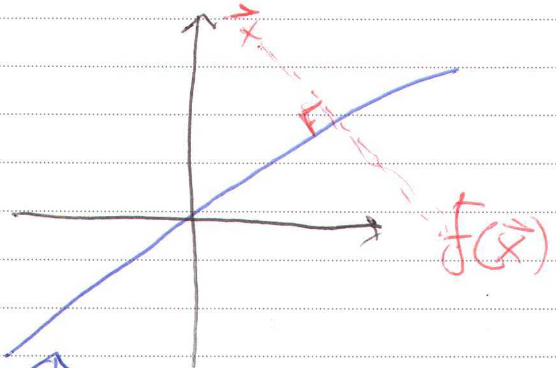
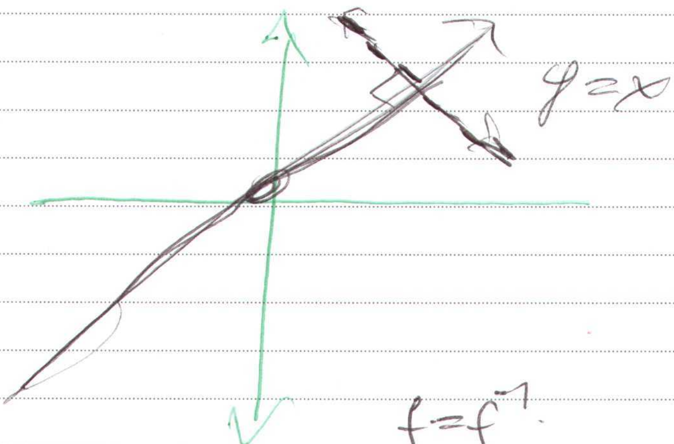


image is $y=x$

if $f: \mathbb{R}^2 \to \mathbb{R}^2$ is surjective the for every elem $\vec{y} \in \mathbb{R}^2 \; \exists \vec{x} \; f(\vec{x}) = \vec{y}$

Is the function a matrix function?



$f = f^{-1}$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ \cancel{x} \end{bmatrix}$$

Surjective & injective

graph $f: \underset{x}{\mathbb{R}} \to \underset{y}{\mathbb{R}}$

plot graph on $\mathbb{R}^2$

$M_{m \times n} \to \{ \text{function} : R^n \to R^m \}$ this function is injective

$f: X \to Y$ function
X is the domain
Y is the co-domain

$f^{-1}(Z) = \{x \in X \mid f(x) \in Z\}$

pre-image of a subset $Z \subseteq Y$

$f: \mathbb{R} \to \mathbb{R}$; $x \mapsto x^2$

$f^{-1}(\{4\}) = \{\pm 2\}$

image pre-image

$= \{x \in \mathbb{R} \mid f(x) = 4\}$

when your function is given algebraically then finding pre-image amounts to ~~solving~~ equations

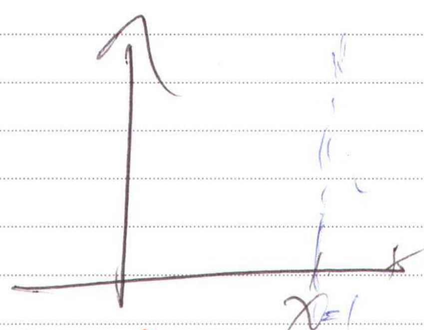
$x^2 = 4$

example #2

$f^{-1}(\{4, 9, 0\}) = \{\pm 2, \pm 3, 0\}$

$f: \mathbb{R}^2 \to \mathbb{R}^2$

projection to x-axis



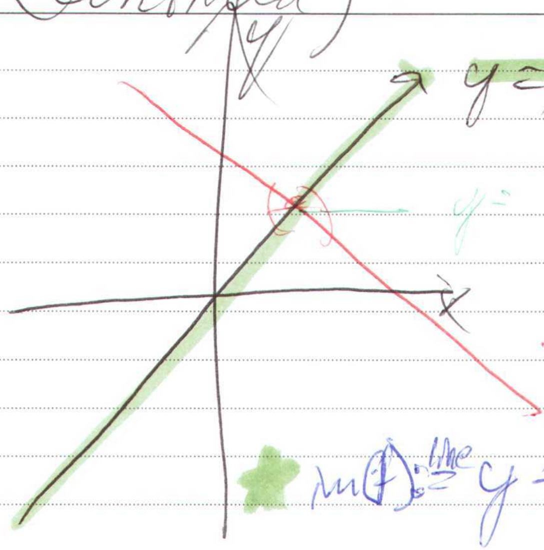
$(x, y) \mapsto (x, 0)$

$f^{-1}(\{1, 1\}) = \emptyset$ DNE

$\boxed{f^{-1}(\{1, 0\}) = \{(1, y) \mid y \in \mathbb{R}\} \text{ the line } x = 1}$

$\operatorname{im}(f) = x\text{-axis}$

(Continued) 9 20 21



$f: \mathbb{R}^2 \to \mathbb{R}^2$ projection to $y = x$

$f^{-1}(\{(1,1)\}) =$ $y = -x+2$ pre-image

$\text{im}(f) =$ the line $y = x$

---

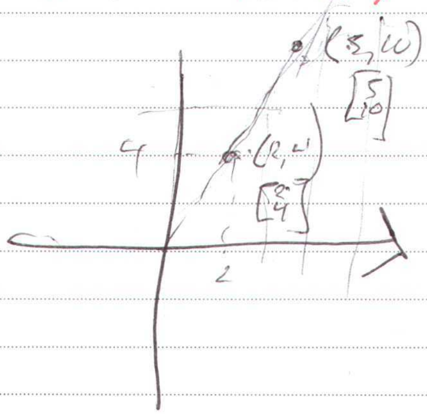
pre image $y \in Y$

$$\text{im}(f) = \{ y \in \mathbb{R} \mid y = x^2 \ \exists x \} = \mathbb{R}_{\geq 0}$$

$A\vec{x} = \vec{b}$ has a soln iff $\vec{b}$ is a linear combination of the column vectors of A

$$\begin{bmatrix} 2 & 5 \\ 4 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x+3y \\ 4x+10y \end{bmatrix} = x\begin{bmatrix} 2 \\ 4 \end{bmatrix} + y\begin{bmatrix} 5 \\ 10 \end{bmatrix} \to$$

$\Rightarrow$ $\text{im}(f) = \{$ linear combination of column vectors $\}$



(continued) 9 20 21

$f: \mathbb{R}^2 \to \mathbb{R}^2$

given $\vec{x} = [x_1, x_2]^T$ how is $f(\vec{x})$ related $\vec{x}$

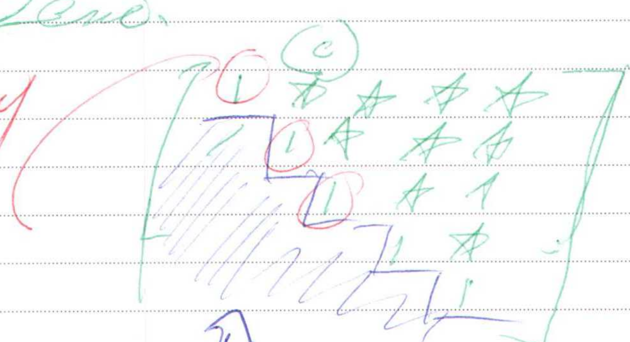
Lecture 7 Row Echelon Forms 9/22/21

Reduced Row echelon Forms: a matrix is said to be (RREF) if: (a) zero rows appear at the bottom (b) first nonzero entry of any row is 1 [leading ones] (c) The nonzeros are placed in an Echelon form. (d) all entries above the leading ones are zero.

$\underbrace{(a) + (b) + (c)}$

Row echelon form not reduced

leading one



(d)

$$\begin{bmatrix} 1 & 0 & * & * & * \\ & 1 & 0 & * & * \\ & & 1 & 0 & * \\ & & & 1 & * \end{bmatrix}$$

This is Reduced

Anything here is zero

If a matrix is not in REF or RREF we can perform ... Elementary Row Operations:

Type I Switch rows
Type II Scale a row
Type III Add rows together

Type I & Type III used in conjugation (e.g. add 2 copies of a row 2 to row 3)

Def: Two matrices are said to be row equivalent if one can be obtained from another via elementary row operations.

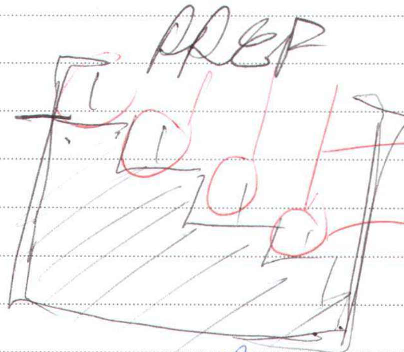
Theorem: Every matrix is row equivalent to one in RREF which is necessarily unique

note: if you only reduce to row echelon form then there are different possible results.

Lecture 8 - RREF and systems of linear equations | 9 | 24 | 21

Recap RREF

echelon



Column of zeros will never cause problem no matter where they are

eg. $\begin{bmatrix} 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is RREF

Definition: a variable is called a free variable if the corresponding column in RREF has no leading one. Otherwise it is a leading variable.

example

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

"w is the free variable and x, y, z are the leading variables"

usually use separate

$$\begin{cases} x + y + z + w = 0 \\ x + 0 + 0\,w = 0 \\ \cancel{x} + 2y + z + 0 = 0 \end{cases}$$

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & \\ \end{array}$$
$$\left[\begin{array}{ccccc|c} 0 & \boxed{1} & 0 & \boxed{2} & 0 & 2 \\ 0 & 0 & \boxed{1} & \boxed{3} & 0 & 3 \\ 0 & 0 & 0 & 0 & \boxed{1} & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\boxed{x_2 + 2x_4 = 2}$

$x_3 + 3x_4 = 3$

$x_5 = 5$

x_1 and x_4 are free variables

Set $x_1 = t$, $x_4 = s$

$x_1 = t$ $x_2 = 2 -$ $x_1 = t$ $x_4 = s$

Two equations $A_1\vec{x} =$
have the same solution iff
are row equivalent

augmented
$[A_1] = [b_1]$

Lecture 9 Elementary Matrices 9/

Elementary row operations can be realized by left multiplication by matrices.

Type I: switch rows
Type II: scale row by non-zero #
Type III: add a multiple of one row to another

Theorem: → Suppose $E_1 E_2 \ldots E_k$ are elementary matrices which reduce a ~~sequence~~ square matrix A to its RREF

Then (a) A is invertible iff RREF(A) = Identity

(b) When this happens $A^{-1} = E_k E_{k-1} \ldots E_1$

Proof: Claim If RREF(A) $\neq I_n$, then A is not invertible

Suppose A is invertible

Then any equation $A\vec{x} = \vec{b}$ has a free variable

$\Rightarrow A\vec{x} = \vec{b}$ either has no solution or has ∞ many solutions contradiction $\Rightarrow$ A is not invertible $\square$

Lecture 10 — 9 2 9

Consider the matrix:

$$A=\begin{bmatrix}3&5\\2&4\end{bmatrix}$$

compute RREF$\left(\begin{bmatrix}A \mid I_n\end{bmatrix}\right)$

$$\left[\begin{array}{cc|cc}3&5&1&0\\2&4&0&1\end{array}\right] \xrightarrow{(1)-(2)} \left[\begin{array}{cc|cc}1&1&1&-1\\2&4&0&1\end{array}\right] \xrightarrow{(2)-2(1)} \left[\begin{array}{cc|cc}1&1&1&-1\\0&2&-2&3\end{array}\right] \to$$

$$\xrightarrow{(2)\times(\frac{1}{2})} \left[\begin{array}{cc|cc}1&1&1&-1\\0&1&-1&\frac{3}{2}\end{array}\right] \xrightarrow{(1)-(2)} \left[\begin{array}{cc|cc}1&0&2&-\frac{5}{2}\\0&1&-1&\frac{3}{2}\end{array}\right]$$

$$\to \frac{1}{2}\begin{bmatrix}4&-5\\-2&3\end{bmatrix} = A^{-1}$$

$$E_1=\begin{bmatrix}1&-1\\0&1\end{bmatrix} \quad E_2=\begin{bmatrix}1&0\\-2&1\end{bmatrix} \quad E_3=\begin{bmatrix}1&0\\0&\frac{1}{2}\end{bmatrix}$$

$$\therefore \left[E_1A \mid E_1I_2\right] \cdots \left[\underbrace{E_4E_3E_2E_1}_{=A^{-1}}A \;\middle|\; \underbrace{E_4E_3E_2E_1I_2}_{A^{-1}}\right]$$

In order to compute A^{-1} we don't need to record individual elem. ma., just append an identity matrix at. & compute RREF.

Lecture 11 Introduction to Determinants — 10/1/2021

Plan: Thoroughly explain 2×2 case + Then Generalize

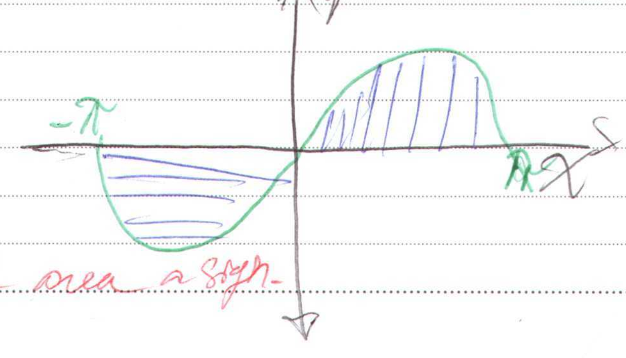
To each $n\times n$ matrix A, we attach a number $\det(A)$ call its determinant.

In other words determinant defines a function

$\det: M_{n\times n} \to \mathbb{R}$

Eg

$$\int_{-\pi}^{\pi} \sin(x)\,dx = 0$$



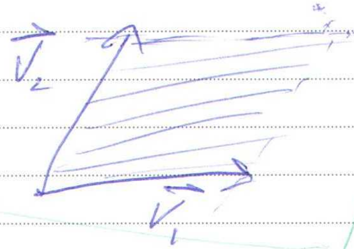
It is natural to give area a sign.

* Always rotate $\vec{v}_1$ to $\vec{v}_2$ - decide if clockwise or counter...

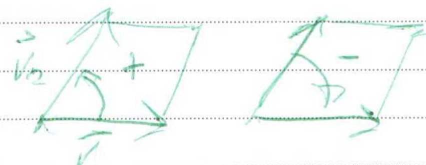
* The abs of shaded area is the magnitude

10 1 21

Given two vectors $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^2$ we ~~define~~ denote by $P(\vec{v}_1, \vec{v}_2)$ the parallelogram defined by $\vec{v}_1$ $\vec{v}_2$



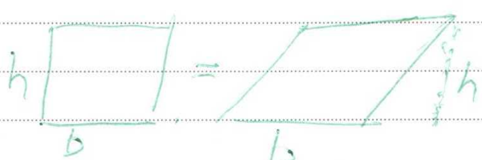
The signed area of $P(\vec{v}_1, \vec{v}_2)$ is 0 if $\vec{v}_1$ and $\vec{v}_2$ are parallel & co-linear
~~#~~ + if $\vec{v}_2$ ~~vector~~ rotates counterclockwise toward $\vec{v}_2$ with $P(\vec{v}_1, \vec{v}_2)$ − if " " clockwise



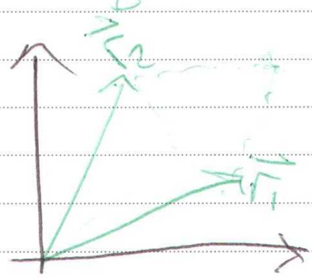
Theorem: For a 2x2 matrix A with row vectors $\vec{v}_1, \vec{v}_2$ (ie. $A = \begin{bmatrix} \cdots \vec{v}_1 \cdots \\ \cdots \vec{v}_2 \cdots \end{bmatrix}$) $\det(A) =$ signed area of $P(\vec{v}_1, \vec{v}_2)$

Example

$A = \begin{bmatrix} 3 & 1 \\ 2 & 3 \end{bmatrix}$ $= \vec{r}_1$, $= \vec{r}_2$

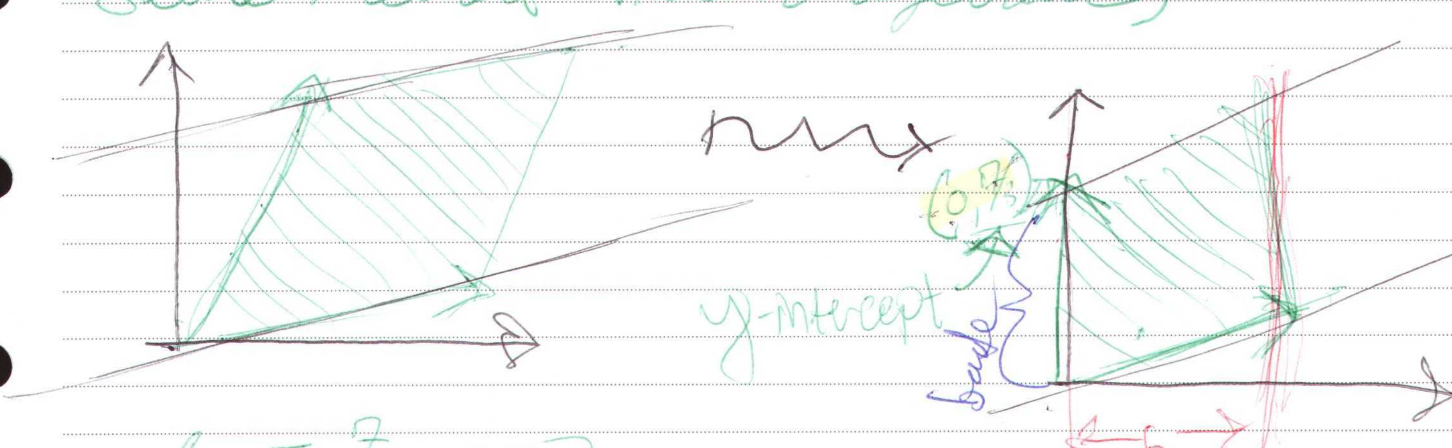


Compute the signed area of $P(\vec{r}_1, \vec{r}_2)$



area $= |\vec{r}_1||\vec{r}_2| \sin\theta$

Slide Method (linear algebraic)



area $= \frac{7}{3} \times 3 = 7$

$\det(A) = 3 \times 3 - 2 \times 1$
$= 7$

$\begin{bmatrix} 3 & 1 \\ 2 & 3 \end{bmatrix} \xrightarrow[E = \begin{bmatrix} 1 & 0 \\ -2/3 & 1 \end{bmatrix}]{② - 2/3 ①} \begin{bmatrix} 3 & 1 \\ 0 & 7/3 \end{bmatrix}$

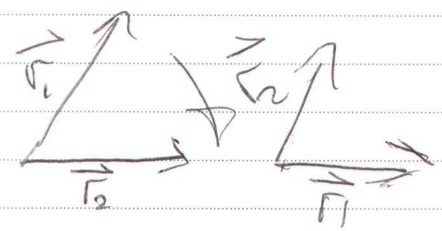
Sliding of the parallegram corresponds to an elementry row operation of Type III

Corollary If E is an elementry matrix of Type II then $\det(EA) = \det(E)\det(A)$

$\det(E) = 1$

Type I

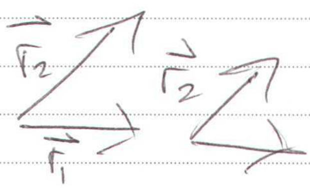
Switch two rows



Type II

Scale ~~by~~ a row ~~a~~ by $\lambda \neq 0$

$\det\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right) = 1$



$\det\left(\begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix}\right) = \lambda$

Theorm

$\forall$ Elem. Matrix: $\det(EA) = \det(E)\det(A)$

suppose that A is not invertible then $\det(A) = 0$

otherwise we can find $E_1 \ldots E_k$ s.t.

$E_k \ldots E_1 \cancel{A} = I_2$

$\det(E_k)\det(E_{k-1}) \ldots$

$\det(A) = \det(I_k) = 1$

Lecture 12 Determinats for $n \times n$ matrices 10/4/2021

→ Last time Given a 2×2 matrix A w/ vectors $\vec{r}_1$, $\vec{r}_2$

$\det(A) =$ Signed area of $P(\vec{r}_1, \vec{r}_2)$

A is pos.

A is neg.

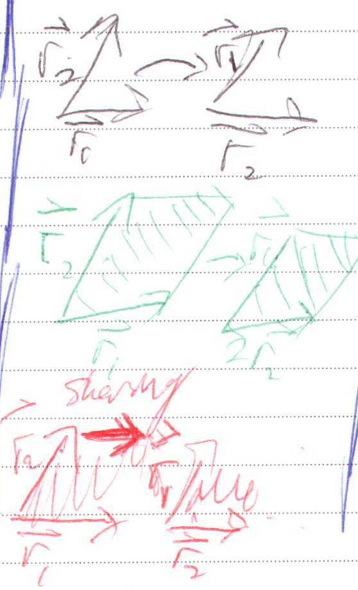
In general the det. for a $n \times n$ matrix A is the signed volume of the parallelogram

$n = 3$

$P(\vec{r}_1, \vec{r}_n)$

(continued) 10 4 2021

Elementary Row operation | Effect of signal

Elementary Row Op.	Effect on area	picture	det of the elem matrix
Type I switch two row	change sign		$\det\begin{pmatrix}0&1\\1&0\end{pmatrix}=-1$
Type II scale a row by $\lambda \neq 0$	multiply by λ scale		$\det\begin{pmatrix}\lambda&0\\0&1\end{pmatrix}=\det\begin{pmatrix}1&0\\0&\lambda\end{pmatrix}=\lambda$
Type III (add a multiple of a row to another)	no effect	shearing	$\det\begin{pmatrix}1&\lambda\\0&1\end{pmatrix}=\det\begin{pmatrix}1&0\\\lambda&1\end{pmatrix}=1$

$\ast$ (base & height remains the same)

$\ast$ Fact: when we go to a higher dim. space the rules on the left still applies

namely, if E is an elementary matrix and A is the next $n\times n$ ~ that $\det(EA)=\det(E)\det(A)$

$\ast$ Fact the determinant of an upper triangular matrix is the product of diagonal entries

$\det=\begin{bmatrix}a&b\\c&d\end{bmatrix}=ad.$ $\det\begin{pmatrix} & \\ 0 & \end{pmatrix}$ multiply diagonal entries

Lecture B Determinants 10/6/21

Recall: For any elem. matrix E we have $\det(A)=\dfrac{\det(EA)}{\det(E)}$

dets $\boxed{\ }$ — all non zero

10 6 21

Notation

Write $|x|$ "two vertical lines" for $\det(x)$

e.g. $\begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} = \det\left(\begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix}\right)$

This is a # not a matrix

Recall $\det = \begin{cases} -1 & \text{Type I} \\ \lambda & \text{Type II} \\ 1 & \text{Type III} \end{cases}$

notice! how we absorb the rule of Type I

Type I $\quad \overset{A}{\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}} = - \overset{B}{\begin{vmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{vmatrix}}$

Type II $\quad \overset{A}{\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 7 & 8 & 9 \end{vmatrix}} = 2 \overset{B}{\begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{vmatrix}}$

$\det(A) = \frac{\det(B)}{\det(E)} = \frac{1}{2}\det(B)$ $\quad \boxed{= 1/2}$

E scales row $\boxed{2}$ by $\boxed{1/2}$

Proof:

$\det(A) = \frac{\det(EA)}{\det(E)} = \frac{\det(E)\det(A)}{\det(E)}$

Type III $\quad \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} \underset{①+②}{=} \begin{vmatrix} 5 & 1 & 9 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$

In practice don't need to remember / write down exactly which operation you did.

Fact: Det. of an upper (lower) triangular matrix = product of diagonal entries

→ Example $\begin{vmatrix} 2 & 2 & 7 \\ 0 & 3 & 9 \\ 0 & 0 & 5 \end{vmatrix} = 2 \begin{vmatrix} 1 & 1 & 7/2 \\ 0 & 3 & 9 \\ 0 & 0 & 5 \end{vmatrix} = 2 \cdot 3 \begin{vmatrix} 1 & 1 & 7/2 \\ 0 & 1 & 3 \\ 0 & 0 & 5 \end{vmatrix}$ →

$= 2 \cdot 3 \cdot 5 \begin{vmatrix} 1 & 1 & 7/2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{vmatrix} = 2 \cdot 3 \cdot 5 \begin{vmatrix} 1 & 1 & 7/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2 \cdot 3 \cdot 5 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2 \cdot 3 \cdot 5$

After you factor out the diagonal entries the rest operations are Type III

(continued)

10 6 24

- Two most important facts about dets.

thm 1 For any two $n \times n$ matrices A, B

$$\det(AB) = \det(A) \cdot \det(B)$$

thm 2 A $n \times n$ matrix A is invertible iff $\det(A) \neq 0$

For two $n \times n$ matrices A, B if one of A and B is non-invertible

then so is AB

$$\det(AB) = \det(A) \det(B) = 0$$

one of these is zero

$A^T\vec{x} = 0$ $\qquad A^T = \begin{bmatrix} 1 & \cancel{-1} 3 & 3 \\ 2 & 4 & 1 \end{bmatrix}$ x_1 x_2 x_3

or

$$\left[\begin{array}{ccc|c} 1 & 7 & 3 & 0 \\ 2 & 4 & 1 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 6 & -5 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 1 & \frac{-5}{6} & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & \frac{13}{6} & 0 \\ 0 & 1 & \frac{-5}{6} & 0 \end{array}\right] \qquad x_1 + \frac{13}{6}x_3 = 0 \qquad \therefore x_2 = \frac{5}{6}t$$

$$x_2 - \frac{5}{6}x_3 = 0$$

Free variable $\quad x_3 = t$

Solution $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t\begin{bmatrix} -13/6 \\ 5/6 \\ 1 \end{bmatrix}$

where $t \in \mathbb{R}$

Let A be a square matrix, then

↳ the minor M_{ij} is the matrix by deleting the i^{th} row and j^{th} column 10/8/21

Eg.

$A = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 5 & 6 \\ 7 & 1 & 2 \end{bmatrix}$ $\quad M_{11} = \begin{bmatrix} 5 & 6 \\ 1 & 2 \end{bmatrix}$ $\quad M_{2,3} = \begin{bmatrix} 3 & -1 \\ 7 & 1 \end{bmatrix}$

$M_{1,2} = \begin{bmatrix} 4 & 6 \\ 7 & 2 \end{bmatrix}$ $\quad M_{22} = \begin{bmatrix} 3 & 2 \\ 7 & 2 \end{bmatrix}$

Definition: The Cofactor @ a_{ij} ((ij)th entry)

is $A_{ij} = (-1)^{i+j} \det(M_{ij})$

assign a sign to $\det(M_{ij})$ in alternating fashion eg. in 3x3 case $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

$A_{11} = +\det(M_{11})$
$A_{12} = -\det(M_{12})$
$A_{22} = +\det(M_{22})$
$A_{23} = -\det(M_{23})$

(Continued) $\det(A) = \det(A^T)$ 10 8 21

Cofactor formula for det.

- Take any row / column
- Take all indices in it
- Sum of $a_{ij} A_{ij}$ for these indices

$\begin{bmatrix} 3 & -1 & 2 \\ 4 & 5 & 6 \\ 7 & 1 & 2 \end{bmatrix}$ expand at 1^{st} row

1^{st} row has indices

$= 3\begin{bmatrix} 5 & 6 \\ 1 & 2 \end{bmatrix} - (-1)\begin{bmatrix} 4 & 6 \\ 7 & 2 \end{bmatrix} + 2\begin{bmatrix} 4 & 5 \\ 7 & 1 \end{bmatrix} = -84$

Relationship between determinants and invertability

(a) Two matrices A, A' are row equivalent $\Leftrightarrow$ $A\vec{x} = 0$ and $A'\vec{x} = 0$ have the same solutions

(b) A homogenous equation $A\vec{x} = 0$ is always consistent ($\vec{x} = 0$ is always a solution) and has ∞ many solutions $\Leftrightarrow$ it has ≥ 1 free variable

(c) If A is a square matrix the following are equivalent

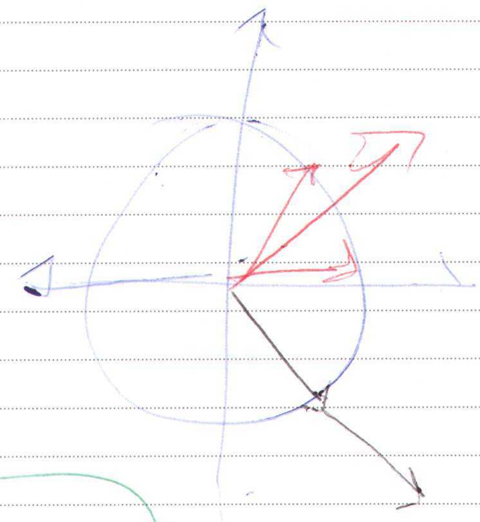
1. $A\vec{x} = 0$ has a free variable
2. RREF(A) has a row of zeros
3. A is not invertible
4. $\det(A) = 0$

Subset that is perpendicular to vector is a subspace

Lecture 16

More on Subspaces

$\left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \;\middle|\; x^2+y^2=1 \right\}$



Exercise is the subset

$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \;\middle|\; 2x+3y+z=6 \right\}$

a subspace of $\mathbb{R}^3$.

Eg $[1, 1, 1]^T \in W$

$2 \cdot [1,1,1]^T \notin W$

$2x+3y+z=12$ W is not closed under [illegible]

Golden rule A subspace has to contain the origin

Reason: Take any $\vec{x} \in W$ = subspace

$\vec{0} = 0 \cdot x \in W$

Is $W' = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \;\middle|\; 2x+3y+z=0 \right\}$ a subspace?

$\vec{v} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \longrightarrow W' = \left\{ x \in \mathbb{R}^3 \;\middle|\; \vec{v} \cdot \vec{x} = 0 \right\}$

Check: closed under addition.

Take $\forall \vec{x}_1, \vec{x}_2 \in W$

Check $\vec{x}_1 + \vec{x}_2 \in W$

$\vec{v} \cdot (\vec{x}_1 + \vec{x}_2) = \underbrace{\vec{v} \cdot \vec{x}_1}_{0} + \underbrace{\vec{v} \cdot \vec{x}_2}_{0} = 0$

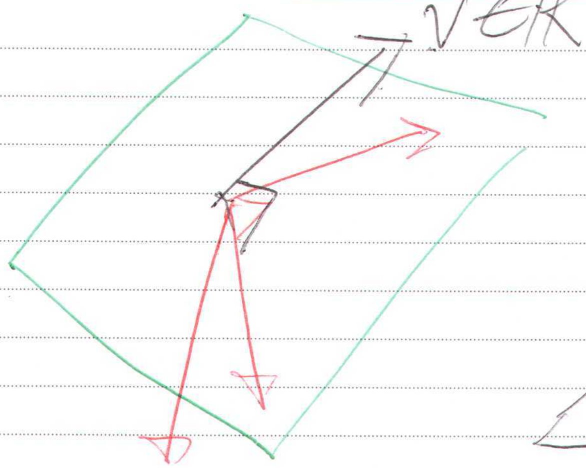
Check closed under [illegible]

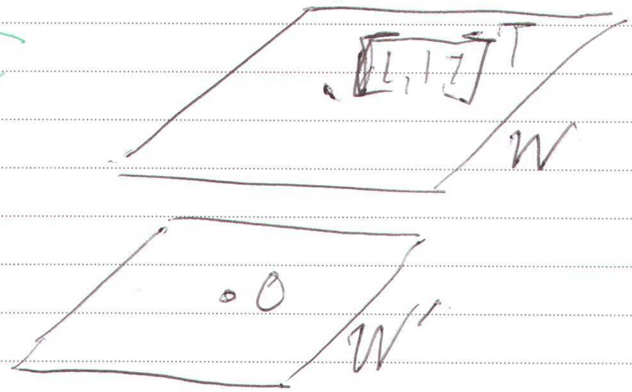
Take $\forall c \in \mathbb{R}$, $\vec{x} \in W$

Check $c\vec{x} \in W$

$\vec{v} \cdot (c \cdot \vec{x}) = c(\vec{v} \cdot \vec{x}) = 0$

TÜL





For any vector space V:

$\{\vec{0}\}$ and V are always subspaces

In $\mathbb{R}^2$: the only other subspaces are lines through $\vec{0}$

In $\mathbb{R}^3$ . . . . . lines/planes $\vec{0}$

Exercise is W a subspace of $M_{2\times 2}$?

$$W = \left\{ \begin{bmatrix} a & b \\ 2b & a \end{bmatrix} a, b \in \mathbb{R} \right\}$$

$$\begin{bmatrix} a_1 & b_1 \\ 2b_1 & a_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ 2b_2 & a_2 \end{bmatrix} = \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ 2(b_1+b_2) & a_1+a_2 \end{bmatrix} \in W \therefore$$

$$c\begin{bmatrix} a & b \\ 2b & a \end{bmatrix} = \begin{bmatrix} ca & cb \\ 2cb & ca \end{bmatrix} \in W \therefore \text{closed under multiplication}$$

closed under addition

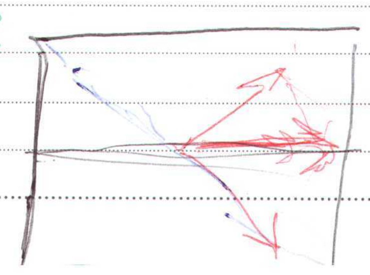
Exercise

Let P_5 be the space of polynomial of degree ≤ 5

$$W_3 = \{ f \in P_5 \mid \deg f \geq 3 \}$$

$0 \in P_5$ and W_3 $\boxed{\text{subspace?}}$

$(3x^3 + 2x^2) + (-3x^3) = 2x^2 \notin W_3$

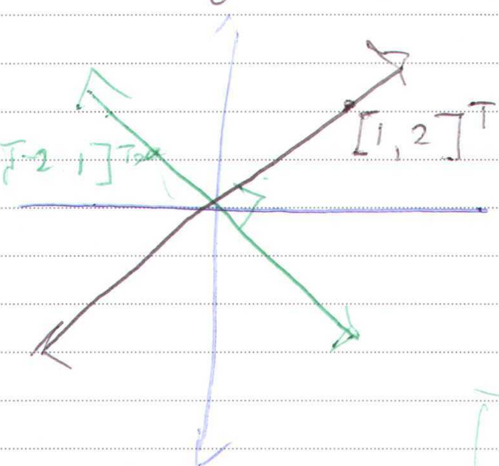


$P_2 \subseteq P_5$ $W_3 = (P_5/P_2) \cup \{0\}$

Subspaces

Analogy: $(\mathbb{R}^2/\mathbb{R}) \cup \{0\}$ Take out P_2 from P_5

Remark! Whenever you talk about vectors, you have to specify the vector space to which your vector belongs (vectors only make sense as elements of vector space)



How to describe the subspace in $\mathbb{R}^2$

lines through $\vec{0}$ all take the form $ax+by=0$

$\boxed{-2x+y=0}$

method 1 orthogonal complement to other line

method 2

$$\left\{ t\begin{bmatrix}1\\2\end{bmatrix} \middle| t \in \mathbb{R} \right\}$$

Lecture 17 Subspaces and Span 10/15/21

Definition: Let V be a vector space and $S \subseteq V$ be a <u>subset</u> (finite or infinite).

The span of S in V is $\text{span}(S) = \{ \vec{v} \in V \mid \exists \vec{v}_1, \dots, \vec{v}_n \in S.\ c_1, \dots, c_n \in \mathbb{R}$ s.t. $\sum_{i=1}^{n} c_i\vec{v_i} = \vec{v} \}$

Span(S) is the <u>subspace</u> of V which <u>consists</u> of linear combinations of element in S. $\sum c_i\vec{v_i} + \sum e_i\vec{v_i} = \sum (c_i+e_i)\vec{v_i}$

also $c\sum c_i\vec{v_i} = \sum (cc_i)\vec{v_i}$

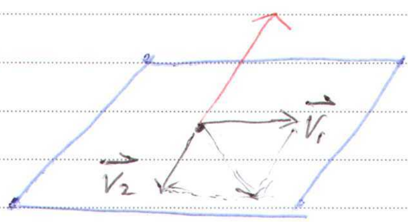
again

$\mathrm{Span}(S) = \mathrm{im}(f_A)$

Eg. $\vec{v} \in V$

$$\mathrm{Span}(\{\vec{v}\}) = \{t\vec{v} \mid t \in \mathbb{R}\}$$

the line which contains $\vec{v}$ (if $\vec{v} \neq 0$)



← $\vec{v}_1 \;\; \vec{v}_2 \in V = \mathbb{R}^3$

$\vec{v}_1 \;\; \vec{v}_2$ ¬colinear

Another POV on Span

$$S = \{\vec{v}_1 \ldots\ldots \vec{v}_m\}$$

Let A be the matrix $\begin{bmatrix} \vdots & \vdots & & \vdots \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \\ \vdots & \vdots & & \vdots \end{bmatrix}$

Let f_A be the matrix transformation given by A

$f_A : \mathbb{R}^m \to \mathbb{R}^n$

$f_A(\vec{e}_i) = \vec{v}_i$

Domain → codomain

eg.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \overset{\vec{e}_2}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}} = \begin{bmatrix} 1\times 0 & -2\times 1 & 3\times 0 \\ 4\times 0 & -5\times 1 & 6\times 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

eg. 2

$\vec{x} = \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix}$ $\quad f_A(\vec{x}) = \sum_{i=1}^{n} c_i \vec{v}_i$

$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 6 \end{bmatrix}$

$\cancel{\vec{x} = \sum_{i=1}^{n} c_i \vec{e}_i}$

Exercise

In $\mathbb{R}^3$ let

$$\vec{V}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \quad \vec{V}_2 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$$

Does the vector $\vec{v} = \begin{bmatrix} 1 \\ 5 \\ -7 \end{bmatrix}$ lie in the ~~image~~ Span $(\{\vec{V}_1, \vec{V}_2\})$

$$\begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ -7 \end{bmatrix}$$

Find

$$\left[A \mid \vec{v}\right] \xrightarrow{RREF} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{array}\right]$$

$\vec{v}$ does lie in the span

$\longrightarrow \vec{v} = 2\vec{v}_1 - 3\vec{v}_2$

$\vec{v} \in Span(\vec{v}_1, \vec{v}_2)$

$\vec{v} \notin Span(\vec{v}_1, \vec{v}_2)$

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{array}\right]$$

$\vec{v}_1 = 2t^2 + t + 2 \quad , \quad \vec{v}_2 = t^2 - 2t$

$\vec{v}_3 = 5t^2 - 5t + 2 \qquad \vec{v}_4 = -t^2 - 3t - 2$

Is $\vec{v} = \boxed{t^2 + t + 2} \in Span(\{\vec{v}_1 \cdots \vec{v}_4\})$

we want to know what $c_1 \cdots c_4$ s.t. $\vec{v} = \sum_{i=1}^{4} c_i \vec{v}_i$

$$t^2 + t + 2 = c_1(2t^2 + t + 2) + c_2(t^2 - 2t) + \cdots$$
$$c_3(5t^2 - 5t + 2) + c_4(-t^2 - 3t - 2)$$

$$\begin{cases} 2c_1 + c_2 + 5c_3 - c_4 = 1 \\ c_1 - 2c_2 - 5c_3 - 3c_4 = 1 \\ 2c_1 \quad + 2c_3 - 2c_4 = 2 \end{cases} \xrightarrow{RREF} \left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$\therefore$ no soln.

(continued)

Definition:

Let $W \subseteq V$ be a subspace. A spanning ~~space~~ set of W is a subset $S \subseteq V$ s.t. $\text{span}(S) = W$

e.g. $\{\vec{e_1}, \vec{e_2}\}$ is a spanning set of $\mathbb{R}^2$

$(\{\vec{e_1}, \vec{e_2}\})$ spans $\mathbb{R}^2$

You cannot find a spanning set which is smaller in size

$\{t^2, t, 1\}$ is a spanning set of P_2

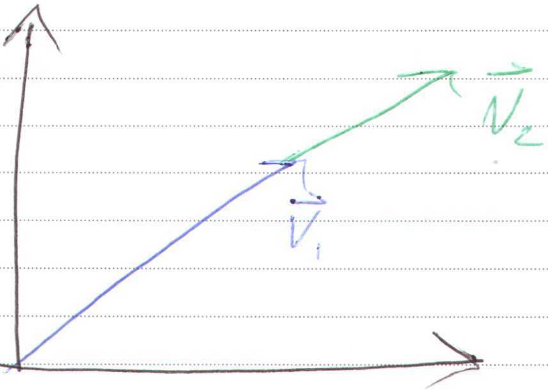
P_2 has dimension $n=3$

Lecture 18 Linear Independence 10/18/2021

Definition: Let $\vec{v}_1, \dots, \vec{v}_m$ be vectors in a vector space V. A linear dependence relation is a equation $\sum_{i=1}^{m} c_i \vec{v}_i = 0$ for some $c_i, \dots, c_m$

understand dimensions of vector space

These vectors are linearly independent if the only linear dependence relation among them is trivial (i.e. $c_i = \dots = c_m = 0$)



Suppose $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

Then $2\vec{v}_1 - \vec{v}_2 = 0$ is a non-trivial lin. dep. rel

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☆☆ Rem: Trivial means all coefficients are 0

(continued) 10 18 21

Recall $\vec{v} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$ then $\vec{v} \in Span(\{\vec{v}_1, \vec{v}_2\})$ but it can be expressed as a linear combination in ∞ many ways

i.e. $\vec{v} = (3-2t)\vec{v}_1 + t \cdot \vec{v}_2$, $t \in \mathbb{R}$

The following two always happen together

1. Every $\vec{v} \in Span(S)$ ($S = \{\vec{v}_1, \cdots \vec{v}_m\}$) can be written as a lin. comb of $\vec{v}_1 \cdots \vec{v}_m$ in more than one way
2. There exist nontrivial linear dependence relation among S

Proof:

1 $\longrightarrow$ 2

prime

Suppose $\vec{v}_1 = \sum_{i=1}^{m} c_i \vec{v}_i = \sum_{i=1}^{m} c_i' \vec{v}_i$ for $\begin{bmatrix} c_i \\ \vdots \\ c_m \end{bmatrix} \neq \begin{bmatrix} c_i' \\ \vdots \\ c_m' \end{bmatrix}$

then $\sum_{i=1}^{m} \underbrace{(c_i - c_i')}_{S_i} \vec{v}_i = 0$ for $\begin{bmatrix} s_i \\ \vdots \\ s_m \end{bmatrix} \neq 0$

$\Rightarrow$ nontrivial lin. depend.

* There is redundancy in the spanning set.

* some vectors in spanning set (S) can be expressed as linear combinations of others.

Exercise $\vec{v}_1 = [3\ 2\ 1]^T$ $\vec{v}_2 = [1\ 2\ 0]^T$ & $\vec{v}_3 = [-1\ 2\ -1]^T$ lin independence?

$$\det(\vec{v}_1, \vec{v}_2, \vec{v}_3) = \begin{vmatrix} 3 & 1 & -1 \\ 2 & 2 & 2 \\ 1 & 0 & -1 \end{vmatrix} = 3 \cdot \begin{vmatrix} 2 & 2 \\ 0 & -1 \end{vmatrix} - 1 \cdot \begin{vmatrix} 2 & 2 \\ 1 & -1 \end{vmatrix} + (-1) \cdot \begin{vmatrix} 2 & 2 \\ 1 & 0 \end{vmatrix}$$

$3(-2+0) - 1(-2-2) - 1(0-2)$

$3(-2) - 1(-4) - 1(-2)$

$-6 + 4 + 2 = 0$

Recall for a square matrix A

$\det(A) = 0$

$\Leftrightarrow A\vec{x} = \vec{0}$ has ∞ many soln.

$\Leftrightarrow$ RREF(A) has a row of zeros $\Leftrightarrow$ A is not invertible

Idem $\det(A) = \det(A^T)$

Exercise Are vectors (in P_2)

$\vec{v}_1$ $3t^2+t-1$, $\vec{v}_2$ $2t^2+2t+2$, $\vec{v}_3$ t^2-1 linearly ind?

$$\cancel{\begin{bmatrix} 3 & 1 & -1 \\ 2 & 2 & 2 \\ 1 & 0 & -1 \end{bmatrix}}^T$$

We want to know if $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = 0$ iff $[0,0,0]^T$ is the only sol. then the system is linearly independent

$$\begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 0 \\ -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$= (3+2+1)c_1 + (1+2)c_2 + (-1+2-1)c_3 = 0$

RREF $\left[\begin{array}{ccc|c} 1 & 0 & 1/2 & 0 \\ 0 & 1 & -1/4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

c_3 free variable

Because c_3 has no leading 1 c_3 has a free variable ∴ ~~the~~ $[000]^T$ is not the only solution → the system is not linearly dependent

---

$\vec{v}_1 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$ $\vec{v}_3 = \begin{bmatrix} 0 & -3 \\ -2 & 1 \end{bmatrix}$

Solution: $t=1$

$c_3 = t$
$c_2 = 2t$
$c_1 = -1$

$\left\{ \begin{bmatrix} -t \\ 2t \\ t \end{bmatrix} t \in \mathbb{R} \right\}$

Span $\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\}$

$$\begin{bmatrix} 2c_1 + c_2 & 1c_1 + 2c_2 - 3c_3 \\ c_2 - 2c_3 & c_1 + c_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2c_1 + c_2 & = & 0 \\ c_1 + 2c_2 - 3c_3 & = & 0 \\ c_2 - 2c_3 & = & 0 \\ c_1 + c_3 & = & 0 \end{bmatrix} = \left[\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 1 & 2 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\vec{v}_1, \vec{v}_2, \vec{v}_3$ are not linearly independent

$\sum_{i=1}^{3} c_i\vec{v}_i = 0$ (in M_{22})

(Continued) 10 20 21

Definition: For any $m \times n$ matrix A the subspace of solutions to $A\vec{x} = \vec{0}$ in $\mathbb{R}^n$ is called the Kernel (null space) of A, denoted by $\text{Ker}(A)$

If $A = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}$ then $A \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \sum_{i=1}^{n} c_i \vec{v}_i$

$\vec{c} \in \text{Ker}(A) \longleftrightarrow$ a lin dep. rel. among $\vec{v}_i$'s

$\text{Ker}(A)$ = Space of linear dependent rel. among column vects of "A"

A row of zeros $\Rightarrow$ implies a free variable only for square matrices

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Lecture 20 Basis And Dimensions 10/22/21

Def^n Let V be a vector space

A subset $S \subseteq V$ is called a basis of V if

- $\text{Span}(S) = V$ $\Rightarrow \forall \vec{v} \in V, \exists \vec{v}_1 \ldots \vec{v}_m \in S$, $c_m \in \mathbb{R}$
- elements of S are linearly independent

unique s.t. $\vec{v} = \sum_{i=1}^{m} c_i v_i$

Example

$\mathbb{R}^n$ has a standard basis

$\{\vec{e}_1 \ldots \vec{e}_m\}$ $\vec{e}_i = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \leftarrow i^{th}$ entry

but $\mathbb{R}^n$ has lots of bases

Exercise prove $S = \{t^2+1,\ t-1,\ 2t+2\}$ is basis for P_2?

$\vec{v}_1 = t^2+1$, $\vec{v}_2 = t-1$, $\vec{v}_3 = 2t+2$

Let $\vec{v} \in P_2$ be a general $\vec{v} = at^2 + bt + c$ $\forall a,b,c \in \mathbb{R}$

we want to know if $\sum^3 a_i \vec{v}_i = \vec{v}$ has a unique soln. $\forall a,b,c \in \mathbb{R}$. Yes-basis No-not basis

$\sum_{i=1}^{3} a_i\vec{v_i} = a_1(t^2+1) + a_2(t-1) + a_3(2t+2$

Continued

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

→ Reduce to RREF

or → Find $\det(A)$

$\det(A) = 1\begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix} = 4$

$\det(A) \neq 0 \Leftrightarrow A\vec{x} = b$ always has a unique soln.

$\det(A) \neq 0 \Leftrightarrow A^{-1}$ exists

$\Leftrightarrow (A\vec{x}=b \Leftrightarrow \vec{x} = A^{-1}b)$

Definition A vector space V is said to be infinite dimensional if it has a finite basis S.
In this case $\dim V = \# S$

Non-example: ① The space of functions $\mathbb{R} \to \mathbb{R}$
② The space of all polynomials are not finite.

Example
- $\dim \mathbb{R}^n = n$
- $\dim P_n = n+1$

a standard basis $\{t^n, t^{n-1}, t, 1\}$

- $\dim M_{mn} = mn$

$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = \sum_{i=1}^{4} a_i\vec{v_i}$

a basis for $M_{m \times n}$ $\{E_{ij} \mid E_{ij}$ is the matrix w/ 1 at (ij)th entry and 0 elsewhere$\}$

t^0 $\vec{v}_1$ $\vec{v}_2$ $\vec{v}_3$ $\vec{v}_4$

Basis $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

10 22 21

Once we have a basis we can talk about coordinance with respect to the other basis.

$S = \{\vec{v}_1 \dots \vec{v}_n\}$ is the basis

$V \longleftrightarrow \mathbb{R}^n \quad \sum_{i=1}^{n} a_i \vec{v}_i \longleftrightarrow \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n$

Remark: A Basis should be thought of as an ordered set when talking an ordered set - when we discuss coordinants.

Exampl

P_2 Basis $\{t^2, t, 1\}$ $\quad \{1, t, t^2\}$

$2t^2+3t+1 \longleftrightarrow \underbrace{\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}}_{\text{coord}} \quad \underbrace{\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}}_{\text{coord}}$

Lecture 21 Basis & Dimensions 10/24/2021

Recall if a vector space V is finite dimensional then we define the dim V to be the number of element in any basis.

WE NEED TO SHOW

~~Pm~~ If S and S' are two different basis of V then # of S = # of S'

Recall coordinals $\quad S = \{\vec{v}_1, \dots \vec{v}_n\}$

$V \xleftrightarrow{S} \mathbb{R}^n$

$\sum_{i=1}^{n} a_i \vec{v}_i \longleftrightarrow \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n \Rightarrow \vec{w}_i \in \mathbb{R}^m = \sum_{j=1}^{n} a_{ij} \vec{v}_j \in \mathbb{R}^n \quad S' = \{\vec{w}_1, \dots \vec{w}_m\}$

Similarly $V \xleftrightarrow{S'} \mathbb{R}^m$

Consider the matrix $A=[a_{ij}]$

Then $A\vec{v}=\vec{w}$ has a Unique Soln. for every $\vec{w}$

$\Rightarrow$ A is invertible

$\Rightarrow$ A is a square matrix

$\Rightarrow m=n$

Exercise For what values of c do the vectors $\vec{V}_1=\begin{bmatrix}1\\0\\-1\end{bmatrix}$ $\vec{V}_2=\begin{bmatrix}2\\1\\1\end{bmatrix}$ and $\vec{V}_3=\begin{bmatrix}1\\1\\c\end{bmatrix}$ in $\mathbb{R}^3$ from a basis for $\mathbb{R}^3$?

Method 1 $A=[\vec{V}_1, \vec{V}_2, \vec{V}_3]$

Fact Thm

If $\dim V=n$, then any n vectors form a basis if and only if they are linearly independent.

Solve $A\vec{x}=0$

Row reduces the Aug. Matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & c-2 & 0 \end{array}\right]$$ Matrix has ∞ many Soln.

Conclusion $c \neq 2$

Method 2 Compute $\det(A)$

$\longrightarrow \det(A) = -1\begin{vmatrix}2 & 1\\ 1 & 1\end{vmatrix} - 1\begin{vmatrix}1 & 1\\ 0 & 1\end{vmatrix} + c\begin{vmatrix}1 & 2\\ 0 & 1\end{vmatrix}$

$= -1 - 1 + c = c-2$

Recall $\det(A)=0 \iff A\vec{v}=\vec{0}$ has ∞ many Solutions

$$A=\begin{bmatrix}1 & 2 & 1\\ 0 & 1 & 1\\ -1 & 1 & c\end{bmatrix}$$

Method 3

$\vec{V}_1$ and $\vec{V}_3$ are linearly independent $\quad \vec{V}_3 \in \text{Span}(\vec{V}_1, \vec{V}_2)$

Suppose $\vec{V}_3 = c_1\vec{V}_1 + c_2\vec{V}_2$

$$\begin{bmatrix}1\\1\\c\end{bmatrix} = c_1\begin{bmatrix}1\\0\\-1\end{bmatrix} + c_2\begin{bmatrix}2\\1\\1\end{bmatrix} \rightarrow \begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}$$

$c_2=1, \ c_1=-1$

$\therefore c \neq 2$

perpendicular $(\vec{n})^{\perp} = \{\vec{v} \in \mathbb{R}^3 \mid \vec{n}\cdot\vec{v} = 0\}$

definition

Exercise

$\vec{v}_1 = \begin{bmatrix}0\\1\\1\end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix}1\\0\\1\end{bmatrix}$ $\vec{v}_3 = \begin{bmatrix}1\\1\\2\end{bmatrix}$

Find dim Span$(\{\vec{v}_1, \vec{v}_2, \vec{v}_3\})$

$\vec{v} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ $\vec{n}\cdot\vec{v} = \begin{bmatrix}1 & -1 & -1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = 0$

Notice $\vec{v}_3 = \vec{v}_1 + \vec{v}_2$ $\therefore$ the Span$(\{v_1, v_2, v_3\}) =$ Span$(\{\vec{v}_1, \vec{v}_2\})$

check for linearly independent dim $= 2$

$A = [\vec{v}_1, \vec{v}_2, \vec{v}_3]$ Ker$(A) =$ Span$\left(\begin{bmatrix}1\\1\\-1\end{bmatrix}\right)$

$\vec{v}_1 + \vec{v}_2 - \vec{v}_3 = 0$

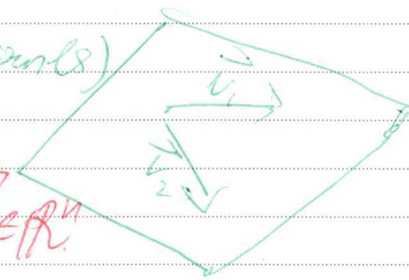
Lecture 22

10/27/2021

Span of anything is always a subspace

Recall if $S = \{\vec{v}_1 \dots \vec{v}_n\}$ is a basis for a vector space V

S = basis for its span(S)



$V \longleftrightarrow \mathbb{R}^n$ $\vec{v} = \sum_{i=1}^{n} a_i\vec{v}_i \longleftrightarrow \begin{bmatrix}a_1\\ \vdots\\ a_n\end{bmatrix} \in \mathbb{R}^n$

we call the vector $\begin{bmatrix}a_1\\ \vdots\\ a_n\end{bmatrix}$ the coord. of $\vec{v} = \sum_{i=1}^{n} a_i\vec{v}_i$ w/ respect to S. denoted by $[\vec{v}]_S$

Exercise Consider the basis S given by $\vec{v}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix}2\\0\\4\end{bmatrix}$ $\vec{v}_3 = \begin{bmatrix}0\\1\\2\end{bmatrix}$ of $\mathbb{R}^3$. Let $\vec{v} = \begin{bmatrix}1\\1\\5\end{bmatrix}$

compute $[\vec{v}]_S$

S-coord

$\begin{bmatrix}\vec{v}_1 & \vec{v}_2 & \vec{v}_3\end{bmatrix}\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix} = \vec{v}$

$\longleftrightarrow \sum_{i=1}^{3} a_i\vec{v}_i = \vec{v}$

Rmk on $\mathbb{R}^3$

$\mathbb{R}^n$ has a standard basis $E = \{\vec{e_1}, \ldots, \vec{e_n}\}$

when we write e.g. $\begin{bmatrix}2\\3\\5\end{bmatrix} \in \mathbb{R}^3$ $\rightarrow 2\vec{e_1}+3\vec{e_2}+5\vec{e_3}$

we are implicitly using E

$$\left[\begin{bmatrix}2\\3\\5\end{bmatrix}\right]_E = \begin{bmatrix}2\\3\\5\end{bmatrix}$$

Question: Suppose $S = \{\vec{v_1}, \ldots, \vec{v_n}\}$

and $T = \{\vec{w_1}, \ldots, \vec{w_n}\}$ are both basis of V.

How can we find a matrix B s.t.

$$\underset{n\times n}{B}\,\underbrace{[\vec{v}]_S}_{\in\mathbb{R}^n} = \underbrace{[\vec{v}]_T}_{\in\mathbb{R}^n}$$

(*)

$$\underbrace{\left[[\vec{v_1}]_E \; [\vec{v_2}]_E \; [\vec{v_3}]_E\right]}_{B}[\vec{v}]_S = [\vec{v}]_E$$

generalize → replace E by general basis T

$$\overbrace{\left[[\vec{v_1}]_T \cdots [\vec{v_n}]_T\right]}^{\parallel\, P_{S\to T}}[\vec{v}]_S = [\vec{v}]_T$$

always square matrix

Transition matrix from S to T denoted by $P_{S\to T}$ (or $P_{T\leftarrow S}$)

$$[\vec{v_i}]_S = \begin{bmatrix}0\\ \vdots \\ 1 \\ \vdots \\ 0\end{bmatrix} = \vec{e_i} \qquad [\vec{v_i}]_W = \vec{e_i}$$

$P_{S\to T}$: $\vec{e_i}$ = ith column of $P_{S\to T} = [\vec{v_i}]_T$

Kernel the ~~set~~ Soln set. $Ax=0$

Let T be a basis for $(\vec{n})^{\perp}$

$T = \left\{ \vec{w}_1 = \begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix}, \vec{w}_2 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \right\}$ Find $P_{T \to S}$

$P_{T \to S} = \left[[\vec{w}_1]_S, [\vec{w}_2]_S \right] = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$

$\vec{w}_1 = 2\vec{v}_1 + 3\vec{v}_2$
$\vec{w}_2 = \vec{v}_1 + 2\vec{v}_2$ $\Rightarrow$ $\begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \vec{w}_1$

Lecture 23 Review 10/19/2021

Recall that lin. alg. centers on solving lin. Eqn. $A\vec{x} = \vec{b}$ $A \in M_{m \times n}$, $\vec{x} \in \mathbb{R}^n$, $\vec{b} \in \mathbb{R}^m$

* What does RREF say: Observe a row in RREF($[A|\vec{b}]$) looks like $\exists c \neq 0$ $[0\ 0 \ldots 0 \mid c]$ then the system is inconsistent.

This means that $A\vec{x} = \vec{b}$ is consistent if there is no such rows.

$\left[\begin{array}{ccc|c} 2 & A & A & a \\ 0 & 1 & A & b \\ 0 & 0 & 3 & c \\ 0 & 0 & 0 & d \end{array}\right]$ If $d=0 \to$ ~~RRA~~ Unique solution b/c there are no free variable

If $d \neq 0$ no soln.

If A has size $m \times n$ and $n > m$ a row of zeros in RREF(A) not imply anything about $\vec{b}$.

Linear Dependence

$\vec{v} = \sum_{i=1}^{n} c_i \vec{v}_i$

Linear Independence

$\mathrm{Ker}(A) = \{\vec{c} \mid A\vec{c} = \vec{0}\}$

$\{\vec{v}_1 \ldots \vec{v}_m\}$ are lin. indep. $\Leftrightarrow$ Ker(A) $= 0$

FACT — $\dim \operatorname{Ker}(A) = \#$ of free variables

The set of all set includes the kernel + some particular soln

Exercise: Find a subset $S' \subseteq S$

$$S = \left\{ \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\1\\4\end{bmatrix}, \begin{bmatrix}0\\1\\2\end{bmatrix}, \begin{bmatrix}-1\\-1\\2\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix} \right\}$$

$$= \vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5$$

Find
$\left\{ \begin{array}{l} \operatorname{Span}(S') = \operatorname{Span}(S) \\ S' \text{ is lin. indep.} \end{array} \right.$ $\longleftrightarrow$ if $\vec{v} \in S$, $\vec{v} \notin S'$ then $\vec{v}$ is a lin. comb of vectors in S'

$A = [\vec{v}_1 \ldots \vec{v}_5]$

$$A \to \begin{bmatrix} 1 & 2 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2/5 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 2 & 0 & 3/5 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2/5 \end{bmatrix}$$

Free variables

Column w/ leading 1's

Take $S' = \{\vec{v}_1, \vec{v}_2, \vec{v}_4\}$

$$\operatorname{Ker}(A) = \left\{ \begin{bmatrix} -2t - 3/5 s \\ t \\ t \\ 2/5 s \\ s \end{bmatrix} \text{ s.t. } t, s \in \mathbb{R} \right\}$$

$$\text{span} = \left(\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3/5 \\ 0 \\ 0 \\ 2/5 \\ 1 \end{bmatrix} \right\} \right)$$

$c_1 \qquad c_2$

$2\vec{v}_1 - \vec{v}_2 = \vec{v}_3 \qquad \vec{v}_5 = 3/5 \vec{v}_1 - 2/5 \vec{v}_4$

TŪL

Ker(A) is the space of lin dep. rel among vectors of A^{11}

$\{a^2t^2+1, at+2, t^2+1\}$ are linearly independent

$$c_1(a^2t^2+1)+c_2(at+2)+c_3(\cancel{t^3}t^2+1)=0$$

$$(c_1a^2t^2+c_1)+(c_2at+2c_2)+(c_3t^2+c_3)=0$$

$$(c_1a^2t^2+c_3t^2)+(c_2at)+(c_1+2c_2+c_3)=0$$

$$(c_1a^2+c_3)t^2+(c_2a)t+(c_1+2c_2+c_3)=0$$

$$\left\{\begin{bmatrix} c_1a^2+c_3 & =0 \\ c_2a & =0 \\ c_1+2c_2+c_3 & =0 \end{bmatrix}\right. \longrightarrow \begin{bmatrix} a^2 & 0 & 1 \\ 0 & a & 0 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\downarrow$$

$$a^2\begin{vmatrix} a & 0 \\ 2 & 1 \end{vmatrix} - 0\begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} + 1\cdot\begin{vmatrix} 0 & a \\ 1 & 2 \end{vmatrix}$$

$$= a^3-a = a(a^2-1)$$

$$a \neq 0, 1, -1$$

Lecture 26 Rank of Matrix

Recall: For a general function $f: X \to Y$ (X, Y are sets)

$$\text{im}(f) = \left\{ y \in Y \,\middle|\, \begin{array}{l} \exists x \in X \\ f(x) = y \end{array} \right\}$$

An $m \times n$ matrix A when viewed as a function $\mathbb{R}^n \to \mathbb{R}^m$ ($\vec{x} \mapsto A\vec{x}$) has an

$$\text{im}(A) = \{ \vec{b} \in \mathbb{R}^m \mid A\vec{x} = \vec{b} \text{ is consistent} \}$$

Prop^n $\text{im}(A)$ is a subspace

check closed under addition $(+)$: $\vec{b}_1, \vec{b}_2 \in \text{im}(A)$

i.e. $A\vec{x}_i = \vec{b}_i$ for some $\vec{x}_i$, $i = 1, 2$

$\Rightarrow \vec{b}_1 + \vec{b}_2 \in \text{im}(A)$ b/c $A(\vec{x}_1 + \vec{x}_2) = \vec{b}_1 + \vec{b}_2$

closed under $(\cdot)$

(Standard Eg.)

in example from Monday $S = \{\vec{v}_1 \dots \vec{v}_5\} \subseteq \mathbb{R}^3$

s.t. S' is a basis for span(S)

$A = [\vec{v}_1 \dots \vec{v}_5]$ $\quad RREF(A) = \begin{bmatrix} 1 & 0 & 2 & 0 & 2 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$
$\qquad\qquad\qquad\qquad\qquad\quad \vec{w}_1 \; \vec{w}_2 \; \vec{w}_3 \; \vec{w}_4 \; \vec{w}_5$

$S' = \{\vec{v}_1, \vec{v}_2, \vec{v}_4\}$ $\qquad$ i.e. $\vec{v}_3 = 2\vec{v}_1 + \vec{v}_2$ $\qquad \vec{v}_4 = 2\vec{v}_1 + \vec{v}_2 + 3\vec{v}_4$

$\rightarrow \text{Span}(S') = \text{Span}(S)$ $\qquad \vec{w}_3 = 2\vec{v} + \vec{v}_2 = 2\vec{w}_1 + \vec{w}_2$

We know that $\vec{v}_1, \vec{v}_2, v_4$ are lin. independ

b/c $\vec{w}_1, \vec{w}_2, \vec{w}_4$ $\qquad$ nullity(A) = dim ker(A)

Rank Span(S) = im(A)

$$A\begin{bmatrix} x_1 \\ \vdots \\ x_5 \end{bmatrix} = \sum_{i=1}^{5} x_i \vec{v}_i$$ S' is a basis for im(A)

What is the Ker(A).

① $\Leftrightarrow -2\vec{v}_1 - \vec{v}_2 + \vec{v}_3 = 0$

$$\begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$ ② $\Leftrightarrow$ $$\begin{bmatrix} -2 \\ -1 \\ 0 \\ -3 \\ 1 \end{bmatrix}$$ These give a basis for Ker A

Rank(A) = # columns w/ leading ones in RREF

nullity(A) = # columns w/o leading ones.

Theorem Rank(A) + nullity(A) = # colums of A

$$\mathbb{R}^n \xrightarrow{A} \mathbb{R}^m$$

$$\dim(\underset{\text{in}}{Ker(A)}) + \dim(\underset{\text{in}}{Im(A)}) = n$$ — subset

Theorem if A is $m\times n$, B is $n\times p$
then
Rank(AB) $\leq$ Rank(A)

Im(AB) = im of im(B) under A

Im(A) = im of $\mathbb{R}^n$ under A

$im(AB) \subseteq im(A)$

$\vec{b} \in Im(AB)$ $\exists \vec{x}$ $AB\vec{x} = \vec{b}$

$\vec{b} \in im(A) \Leftarrow A(B\vec{x}) = \vec{b}$

$$\begin{tikzcd} \mathbb{R}^p \arrow[r, "B"] \arrow[rr, bend left, "AB"] & \mathbb{R}^n \arrow[r, "A"] & \mathbb{R}^m \end{tikzcd}$$

$\vec{x} \longmapsto B\vec{x} \longmapsto A(B\vec{x}) = (AB)\vec{x}$

matrix product
$m\times n$ $n\times p$
[need result]

Theorem: Let A, B be as before

$\text{Rank}(AB) \leq \text{Rank}(B)$

$\dim \text{im}(AB) \leq \dim \text{im}(B)$

Claim $\ker(AB) \supseteq \ker(B)$

if $B\vec{x} = \vec{0}$ then

$(AB)\vec{x} = A(B\vec{x}) = A\vec{0} = \vec{0}$

$\Rightarrow \text{nullity}(AB) \geq \text{nullity}(B)$

$\text{Rank}(B) + \text{nullity}(B) = p = \text{rank}(AB) + \text{nullity}(AB)$

$\Rightarrow$

$\text{rank}(B) \geq \text{rank}(AB)$

$W \subseteq V \qquad \dim(W) \leq \dim V$ ✓

Lecture 27 Linear Transformation 11/17/2021

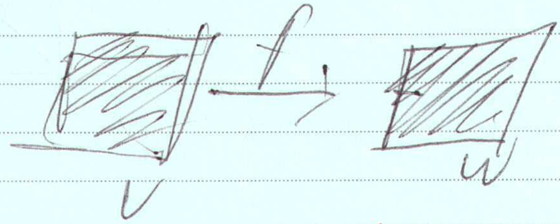
Recall: Every $m \times n$ matrix A defines a function $\mathbb{R}^n \to \mathbb{R}^m$
$f_A(\vec{x}) = A\vec{x}$

Question! Among all functions $\mathbb{R}^n \to \mathbb{R}^m$ which of these come from matrices?

$\to$ Main Theorem!
{ f is a matrix transformation iff it is a linear transformation

If $f = f_A$ for some A, then f preserves
- addition $(f(\vec{x} + \vec{y}) = f(\vec{x}) + f(\vec{y}))$
- scalar multiply $(f(\lambda\vec{x}) = \lambda \cdot f(\vec{x}), \forall \vec{x})$
  $A(\lambda\vec{x}) = \lambda(A\vec{x})$

Def 2: Given vector spaces V and W. A function $f: V \to W$ is called a linear transformation if it preserves addition and scalar multiplication.



Question: If we know $f = f_A$ how to recover A from f?

use standard basis $A\vec{v} = f$ $\quad A = [f(\vec{e_1}) \cdots f(\vec{e_n})]$

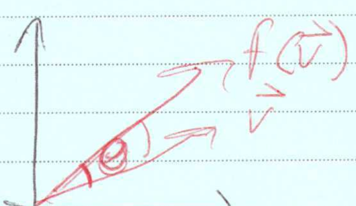
Recall the i^{th} column vector A is $A\vec{e_i} = f(\vec{e_i})$

For any $\vec{v} \in R$ express $\vec{v} = \sum_{i=1}^{n} \lambda_i \vec{e_i}$

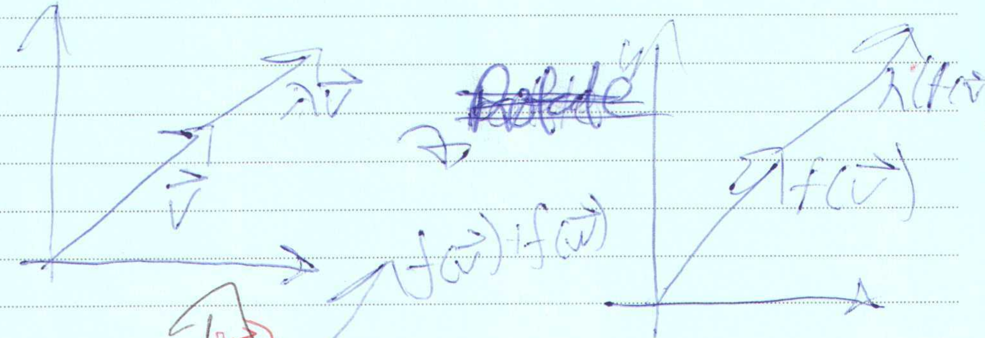
$$f(\vec{v}) = f\left(\sum_{i=1}^{n} \lambda_i \vec{e_i}\right) = \sum_{i=1}^{n} \lambda_i f(\vec{e_i}) = \sum_{i=1}^{n} \lambda_i A\vec{e_i} \quad \square$$

f is a linear transformation $= A\vec{v}$ $\square$

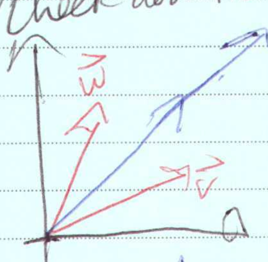
Example: Let $f: R^2 \to R^2$ be rotation by angle θ



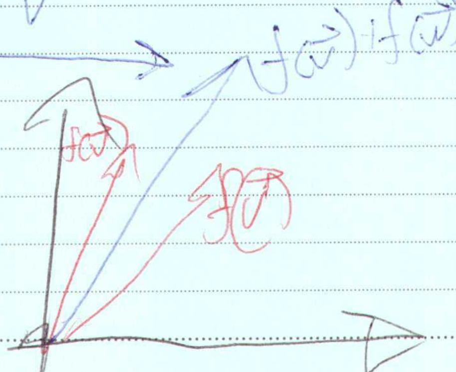
1) Check if it preserves scalar multiplication



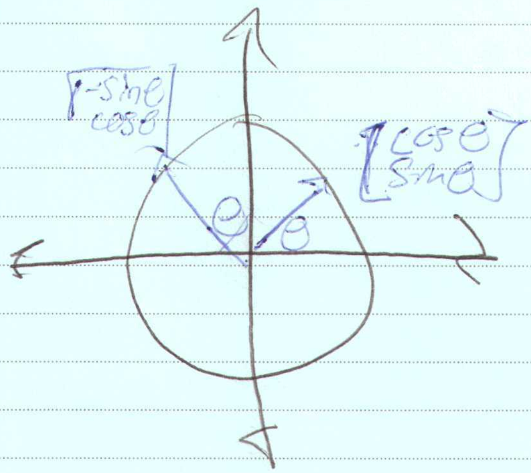
2) Check addition



↳ rotate



Find the matrix A
s.t. $f = f_A$

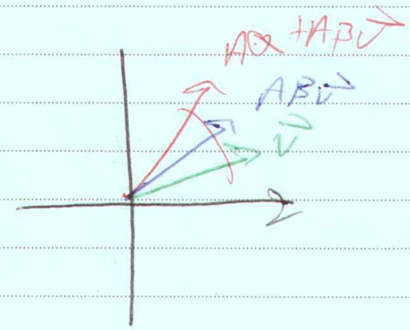


$A = [\, f(\vec{e}_1) \; f(\vec{e}_2) \,] =$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Express $\sin(\alpha + \beta)$ on terms of sin, cos of $\alpha + \beta$

$A_{\alpha + \beta} = A_\alpha + A_\beta$



$$\begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix} =$$

$$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$

Lecture 29 - Lin. Transformation Continued

Recall: If $L: V \to W$ is a lin trans, and we've chosen bases, $S = \{\vec{v}_1 \dots \vec{v}_n\}$ for V and $T = \{\vec{w}_1 \dots \vec{w}_m\}$ for W. Then we have the matrix of L w.r.t S and T

$$\begin{bmatrix} [L(\vec{v}_1)]_T & \cdots & [L(\vec{v}_n)]_T \end{bmatrix}$$ w/ the size $\dim W \times \dim V$

Example Let $V \subseteq C^\infty$ be the subspace w/ basis $S = \{xe^x, e^x, x, 1\}$

Let $L: V \to V$ be $f \to f'$

Find the matrix of L w.r.t. S

$L(xe^x) = xe^x + e^x$
$L(e^x) = e^x$
$L(x) = 1$
$L(1) = 0$

$$\begin{array}{c|cccc} & L(xe^x) & L(e^x) & L(x) & L(1) \\ \hline xe^x & 1 & 0 & 0 & 0 \\ e^x & 1 & 1 & 0 & 0 \\ x & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{array}$$

Def^n $L: V \to W$ be a lin. trans

$\ker(L) = \{\vec{v} \in V \mid L(\vec{v}) = \vec{0}\} \subseteq V$

$\operatorname{im}(L) = \{\vec{w} \in W \mid \exists \vec{v} \in V \text{ w/ } L(\vec{v}) = \vec{w}\} \subseteq W$

$\operatorname{Rank}(L) = \dim \operatorname{im}(L)$ $\quad$ $\operatorname{nullity}(L) = \dim \ker(L)$

Def^n L is said to be

- (one-to-one) injective if $L(x) = L(y) \leftrightarrow x = y$
- (onto) Surjective if $W = \operatorname{im}(L)$

Also L is injective $\longleftrightarrow$ $\ker(L) = 0$

Ker(A) tells you $L(f) = f'$

Ker (L) in S-coord If $L(f)=0$, $f' \equiv 0$ In other words $f \in \text{Span}\{1\}$

f is a constant.

What is the basis for im(L)?

$\Rightarrow \{xe^x, e^x, 1\} \to RREF(A) = \begin{array}{cccc} 1^{st} & 2^{nd} & 3^{rd} & \\ \left[\begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \end{array}\right. & \begin{array}{c} 0 \\ 1 \\ 0 \\ 0 \end{array} & \begin{array}{c} 0 \\ 0 \\ 1 \\ 0 \end{array} & \left.\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array}\right] \\ xe^x+e^x & e^x & 1 & \end{array}$

Columns 1^{st} 2^{nd} 3^{rd} form a basis for im(A)

Consider

$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 0 \\ 0 & 0 \end{bmatrix}$ $RREF(A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ $im(A) \neq im(RREF(A))$

$Ker(A) = Ker(RREF(A))$

The rank - nullity theorem works for lin trans $L: V \to W$ i.e. $rank(L) + nullity = \dim V$

by $L: P_3 \to P_3$ given $L(f(t)) = f(t-1)$

$S = \{t^3, t^2, t, 1\}$

$L(t^3) = t^3 - 3t^2 + 3t - 1$
$L(t^2) = t^2 - 2t + 1$
$L(t) = t - 1$
$L(1) = 1$

$\begin{array}{cccc} L(t^3) & L(t^2) & L(t) & L(1) \\ \left[\begin{array}{c} 1 \\ -3 \\ 3 \\ -1 \end{array}\right. & \begin{array}{c} 0 \\ 1 \\ -2 \\ 1 \end{array} & \begin{array}{c} 0 \\ 0 \\ 1 \\ -1 \end{array} & \left.\begin{array}{c} 0 \\ 0 \\ 0 \\ 1 \end{array}\right] \end{array}$

Invertible? yes

Lecture 30 Change of Bases | 11 15 01

$L: V \to W$ by choosing bases for V and W and representing L by a matrix w r t the chosen bases

- If we use bases $S = \{\vec{v}_1 \cdots \vec{v}_n\}$ and $T = \{\vec{w}_1 \cdots \vec{w}_m\}$ for V, W respectively then the matrix $[L]_{ST} =$

$$\begin{bmatrix} [L(\vec{v}_1)]_T & \cdots & [L(\vec{v}_n)]_T \end{bmatrix}$$

Exercise: Let $L: R^3 \to R^2$ be given by

$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$ Let $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

and $T = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$

Find the matrix representation L wrt S & T

$$[L(\vec{v}_1)]_T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$[L(\vec{v}_2)]_T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

Recall $[L(\vec{v}_1)]_T = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ means $c_1 \vec{w}_1 + c_2 \vec{w}_2 = L(\vec{v}_1)$

$$\begin{bmatrix} \vec{w}_1 & \vec{w}_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = L(\vec{v}_1) - [L(\vec{v}_1)]_T = [\vec{w}_1, \vec{w}_2]^{-1} L(\vec{v}_1) = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$P_{T \to E}$ T-coord Standard coord

1st column

$L(\vec{v_2})_T$

$= \left[\vec{w}_1, \vec{w}_2\right]^{-1} L(\vec{v}_2)$

$= \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

2nd Column

$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ $\begin{matrix} 2-3 \\ -1+3 \end{matrix}$ $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$\frac{1}{2-1} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

~~$= \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2+1 \\ 3-3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$~~

Ans: $\begin{bmatrix} 3 & 1 & \cancel{8} \\ -1 & 1 & \cancel{8} \end{bmatrix} = \begin{bmatrix} 3 & 1 & -1 \\ -1 & 1 & 2 \end{bmatrix}$

If S & T are both bases for the same vector space V

$P_{S \to T} = \left[[\vec{v}_1]_T \cdots\cdots [\vec{v}_n]_T \right]$ $\quad S = \{\vec{v}_1 \cdots\cdots \vec{v}_n\}$

Exercise

Let $V \leq \mathbb{R}^3$ be the plane of bases

Find $P_{S \to T}$ and $P_{T \to S}$

$S = \left\{ \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \right\}$

$T = \left\{ \vec{w}_1 = \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix}, \vec{w}_2 = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \right\}$

$P_{T \to S} = \left[[\vec{w}_1]_S \; [\vec{w}_2]_S \right]$

$\left[\vec{v}_1, \vec{v}_2\right] [\vec{w}_1]_S = \vec{w}_1 \to \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 3 \end{bmatrix} [\vec{w}_1]_S = \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix}$

$[\vec{w}_1]_S = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \iff 2\vec{v}_1 + \vec{v}_2 = \vec{w}_1$

Similarly $[\vec{w}_2]_S = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \iff \vec{v}_1 + \vec{v}_2 = \vec{w}_2$

$P_{T \to S} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ $\quad P_{S \to T} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-1}$

$= \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$

TÜL ① Coordinate ②

Let $L: V \to W$ be a lin transformation

Let S, S' be a bases for V

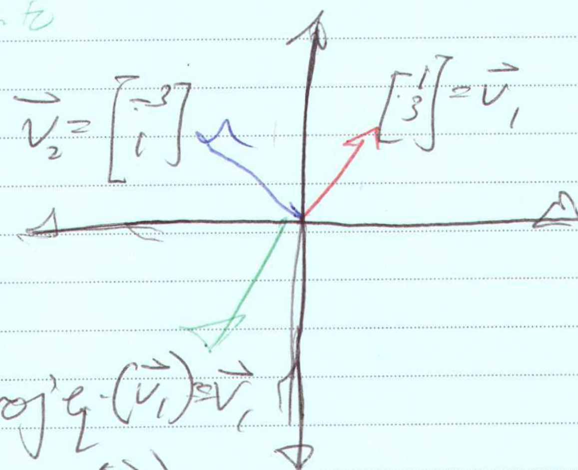
T, T' ----- for W

Suppose we know that $[L]_{ST}$

$$[L]_{S'T'} = P_{T \to T'} [L]_{ST} P_{S' \to S}$$

Example: Suppose we want to find the matrix rep. $proj_L$ in std coords

$L = l_n \quad y = 3x$ — $proj_L$ = Orthogonal Projection to L



$$\left[proj_L\right]_S = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$proj_L(\vec{v}_1) = \vec{v}_1$

$$\begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}^{-1}$$

$proj_L(\vec{v}_2) = \vec{0}$

$P_{S \to E}$

$$[proj_L]_E = P_{S \to E} [proj_L]_S P_{E \to S}$$

$$\begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}^{-1}$$

Lecture 31 Change of Basis

Rem:
- $L: V \to W$

S, S' bases for V

T; T' .... W

$$[L]_{S',T'} = \underbrace{P_{T\to T'}[L]_{S,T}P_{S'\to S}}$$

"Pf"

$$\underbrace{[L]_{S',T'}}_{\text{matrix size } \dim W \times \dim V}[\vec{v}]_{S'} = [L(\vec{v})]_{T'}$$

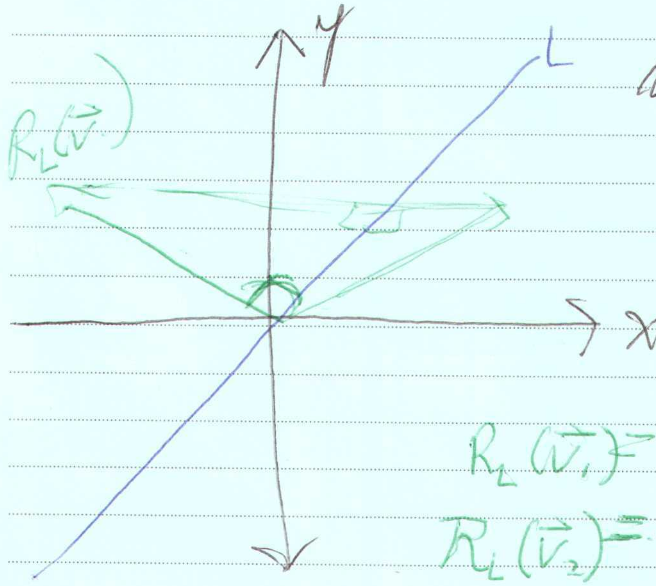
$$P_{S'\to S}[\vec{v}]_{S'} = [\vec{v}]_S$$

$$[L]_{S,T}([\vec{v}]_S) = [L(\vec{v})]_T$$

$$P_{T\to T'}([L(\vec{v})]_T) = [L(\vec{v})]_{T'}$$

Let $R_L: \mathbb{R}^2 \to \mathbb{R}^3$ be a reflection across L

$L = \{y = 3x\}$



Find the matrix for L wrt std. basis

$R_L(\vec{v}_1) = \vec{v}_1$ $\quad S = \{\vec{v}_1, \vec{v}_2\}$

$R_L(\vec{v}_2) = -\vec{v}_2$ $\quad$ new basis for $\mathbb{R}^2$

$$[R_L]_S = \left[[R_L(\vec{v}_1)]_S\,[R_L(\vec{v}_2)]_S\right] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$[R_L]_E = \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 3/5 \\ 3/5 & 1 \end{bmatrix}$$

If $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{v}$ then $[\vec{v}]_S = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$\boxed{\left[\vec{v}_1, \vec{v}_2\right]} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \vec{v}$

$P_{S \to E}$ $\quad$ $[\vec{v}]_S$

$P_{E \to S} = P^{-1}_{S \to E}$

E = std basis

Let A be a matrix B, C, be invertible matrices of suitable sizes.

Then $\text{rank}(BAC) = \text{rank}(A)$

$\text{nullity}(BAC) = \text{nullity}(A)$

Say A is $m \times n$. Then it defines a lin transition $L: \mathbb{R}^n \to \mathbb{R}^m$

in other words A rep L in std bases. Any invertible matrix can be seen as a change of bases matrix.

So: BAC rep L wrt some other bases

$\text{rank}(BAC) = \dim \text{Im}(L) = \text{rank}(A)$

Same applies to nullity.

$\mathbb{R}^n \to \mathbb{R}^n$

Recall before we talked about determinants of $n \times n$ matrix

Def Let $L: V \to V$ If $S = \{\vec{v}_1, \dots, \vec{v}_n\}$ is any basis of V, then $\det(L) = \det([L]_S)$

If T is another basis

$[L]_T = P_{S\to T} [L]_S$

If T is another basis $[L]_T = P_{S\to T}[L]_S P_{S\to T}^{-1}$

$\det([L]_T) = \cancel{\det(P_{S\to T})} \det([L]_S) \cancel{\det(P_{S\to T})^{-1}}$

Defn: $L: V \to V$ is invertible if there exists a linear transformation $L^{-1}: V \to V$

s.t. $L \circ L^{-1} = L^{-1} \circ L = id_V$

$id_V(\vec{v}) = \vec{v}$ (← does nothing)

∴ represented by the identity matrix wrt any basis S of V

Prop: L is invertible iff Pf: choose a basis S and reduce to the case for matrices

T/F Let L be any line $\mathbb{R}^2$, R_L = reflection not L

$\det(R_L) = -1$

True R_L is rep $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

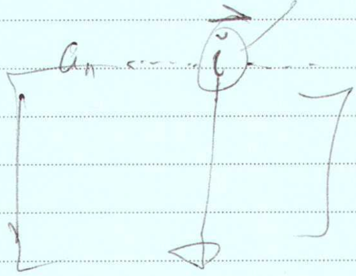
$\det(R_L) = -1$

Lecture 32 Let $L: V \to V$ be a lin. trans.

$S = \{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis

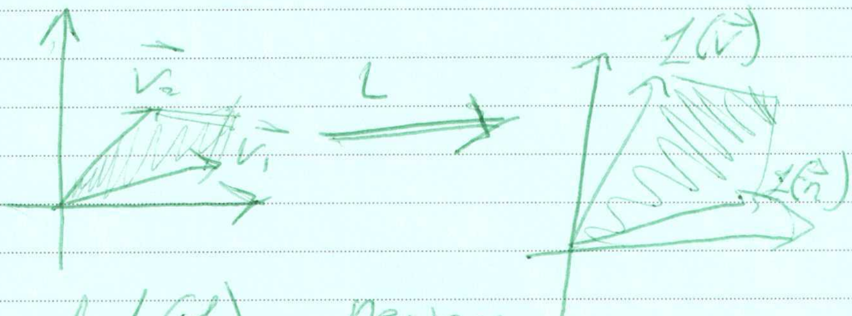
$\det(L) = \det([L]_S)$

$\vec{v}_i = L_A(\vec{e}_i)$



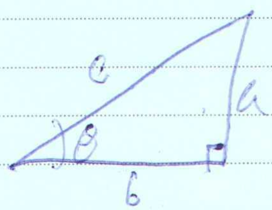
If $L: V \to V$ is a general formula

$$\det(L) = \frac{vol(P(L(\vec{v}_1) \dots L(\vec{v}_n)))}{vol(P(\vec{v}_1, \dots \vec{v}_n))}$$



$$\det(L) = \frac{\text{new area}}{\text{old area}}$$

Example



$a^2 + b^2 = c^2$

$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$$

$$\sin^2\theta + \cos^2\theta = 1$$

Consider

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

"Rotations of every vector in $\mathbb{R}^2$ by θ"

$$\begin{tikzcd}
{[\vec{v}]_E} \arrow[r, mapsto, "P_{E\to T}"] \arrow[d, "{[L]_E}"'] & {[\vec{v}]_T} \arrow[d, "{[L]_T}"] \\
{[L(\vec{v})]_E} & {[L(\vec{v})]_T} \arrow[l, mapsto, "P_{T\to E}"]
\end{tikzcd}$$

Let V and W be vector spaces

Let $\mathrm{Hom}(V,W)$ denote the space of lin. trans from V to W

When $V = \mathbb{R}^n$, $\mathbb{R}^m$ we have a natural identification

$$\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m) = M_{m\times n}$$

space of $m\times n$ matrices.)

i.e. all lin trans: $\mathbb{R}^n \to \mathbb{R}^m$ are matrix transformations

11 22 21

$\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m) \longleftrightarrow M_{m\times n}$

$L \longleftrightarrow A = \begin{bmatrix} \vdots & & \vdots \\ L(\vec{e}_1) & \cdots & L(\vec{e}_n) \\ \vdots & & \vdots \end{bmatrix}$

Every $\vec{v} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n$ is equal to $\sum_{i=1}^{n} c_i \vec{e}_i$

Remark A is the matrix rep L wrt Std Bases

we need $L(\vec{v}) = A\vec{v} \quad \forall \vec{v} \in \mathbb{R}^n$

$L\left(\sum_{i=1}^{n} c_i \vec{e}_i\right) = \sum_{i=1}^{n} c_i L(\vec{e}_i) = A\vec{v}$

$\begin{bmatrix} \vdots & & \vdots \\ \vec{v}_1 & \cdots & \vec{v}_n \\ \vdots & & \vdots \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \sum_{i=1}^{n} c_i \vec{v}_i$

identity

$id_V : V \to V$ does nothing

$id_V(\vec{v}) = \vec{v}$

Let $L: V \to W$ i.e. ($L \in \mathrm{Hom}(V, W)$)

L is invertible if $\exists L' : W \to V$
s.t. $L \circ L' = id_W$, $L' \circ L = id_V$

L' undoes L in the sense $L(L'(\vec{w})) = \vec{w}$

If L' exists then it is unique and we denote it by L^{-1}.

Continued

(L, L^{-1}) is just a generalization of (A, A^{-1})

A = invertible matrix

✓ L is invertible

✓ L is one-to-one (injective)

✓ L is onto (surjective)

For a lin. trans. $L: V \to W$

one-to-one $\Longleftrightarrow$ $\ker(L) = 0$

onto $\Longleftrightarrow$ $\mathrm{im}(L) = W$

For any vector space V ~~$\mathrm{Hom}(\mathbb{R}^n, V)$~~

$\mathrm{Hom}(\mathbb{R}^n, V) \Longleftrightarrow \left\{ \begin{matrix} \text{ordered sets} \\ \text{of } n\text{-vectors in } V \end{matrix} \right\}$

$L_T: \mathbb{R}^n \to V$

$L_T(\vec{e}_i) = \vec{v}_i$ $\qquad T = \{\vec{v}_1, \dots, \vec{v}_n\}$ $\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in \ker(L_T)$

$\Longleftrightarrow$

$$L_T\left(\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}\right) = \sum_{i=1}^{n} c_i L(\vec{e}_i) = 0$$

$$= \sum_{i=1}^{n} c_i \vec{v}_i$$

$im(L_T) = Span(T)$

$$L_T\left(\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}\right) = \sum_{i=1}^{n} c_i \vec{v_i}$$

$$\left\{\begin{array}{l} T \text{ is lin indep} \overset{\text{iff}}{\Longleftrightarrow} \cancel{\ker}(L_T) \text{ is one-to-one} \\ Span(T) = im\, V \Longleftrightarrow L_T \text{ is onto} \\ T \text{ is a basis for } V \Longleftrightarrow L_T \text{ is invertible} \end{array}\right.$$

$$\forall \vec{v} \in V, \ \underbrace{[\vec{v}]_T}_{R^n} = L_T^{-1}(\vec{v})$$

Matrices Representing lin transformation

Let $L: V \to W$ be a lin. trans
Let S & T be bases for V and W

$[L]_{ST} =$

Suppose $S = \{\vec{v_1} \dots \vec{v_n}\}$ ^ $T = \{\vec{w_1} \dots \vec{w_m}\}$

$$\text{WTS } [L]_{ST} \underbrace{[\vec{v}]_S}_{R^n} = \underbrace{[L(\vec{v})]_T}_{R^m}$$

$$\begin{tikzcd} V \arrow[r, "L"] & W \\ R^n \arrow[u, "L_S"] \arrow[r, "{[L]_{ST}}"] & R^m \arrow[u, "L_T"'] \end{tikzcd}
\qquad
\begin{tikzcd} V \arrow[r, "L"] & W \arrow[d, "L_T^{-1}"] \\ R^n \arrow[u, "L_S"] \arrow[r] & R^m \end{tikzcd}$$

Let V be a vector space

Let S, S' be two bases

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "L_S"] \arrow[rr, bend right, "P_{S,S'}"'] & V \arrow[r, "L_{S'}^{-1}"] & \mathbb{R}^n \end{tikzcd}$$

$$P_{S,S'} \underbrace{[\vec{v}]_S}_{L_S^{-1}(\vec{v})} = \underbrace{[\vec{v}]_{S'}}_{L_{S'}^{-1}(\vec{v})}$$

$$\Rightarrow P_{SS'} \circ L_S^{-1} = L_{S'}^{-1}$$

$$[L]_{S,T} = P_{T',T}[L]_{S',T'} P_{S,S'}$$

$$\Rightarrow P_{S,S'} = L_{S'}^{-1} \circ L_S$$

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "{[L]_{S,T}}"] \arrow[dd, bend right=60, "P_{S,S'}"'] & \mathbb{R}^m \arrow[d, "L_T"] \arrow[dd, bend left=60, "P_{T,T'}"] \\ V \arrow[u, "L_S^{-1}"] \arrow[r, "L"] \arrow[d, "L_{S'}^{-1}"'] & W \\ \mathbb{R}^n \arrow[r, "{[L]_{S',T'}}"] & \mathbb{R}^m \arrow[u, "L_{T'}"'] \end{tikzcd}$$

$$\Rightarrow [L]_{S,T} = \underbrace{P_{T',T}}_{P_{T,T'}^{-1}} [L]_{S',T'} P_{S,S'}$$

Let $\mathbb{R}^2 \to P_1$ be lin T defnd. b

$$L\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = (-x_1+x_2)t + (2x_1+x_2)$$

where E = std. basis for $\mathbb{R}^2$

Compute $[L]_{ET}$ $\qquad T = \{t+1,\ 2t+1\}$ $\qquad \vec{e}_1 = \begin{bmatrix}1\\0\end{bmatrix}$ $\quad \vec{e}_2 = \begin{bmatrix}0\\1\end{bmatrix}$

① $L(\vec{e}_1) = -t+2 \qquad L(\vec{e}_2) = t+1$

we want to express $(-t+2)$ as a lin combination

$$(-t+2) = c_1(t+1) + c_2(2t+1)$$

↑ Solve

$$[L(\vec{e}_1)]_T = \begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}5\\-3\end{bmatrix}$$

$$[L(\vec{e}_2)]_T = \begin{bmatrix}1\\0\end{bmatrix} \qquad \therefore [L]_{ET} = \begin{bmatrix}5 & 1\\-3 & 0\end{bmatrix}$$

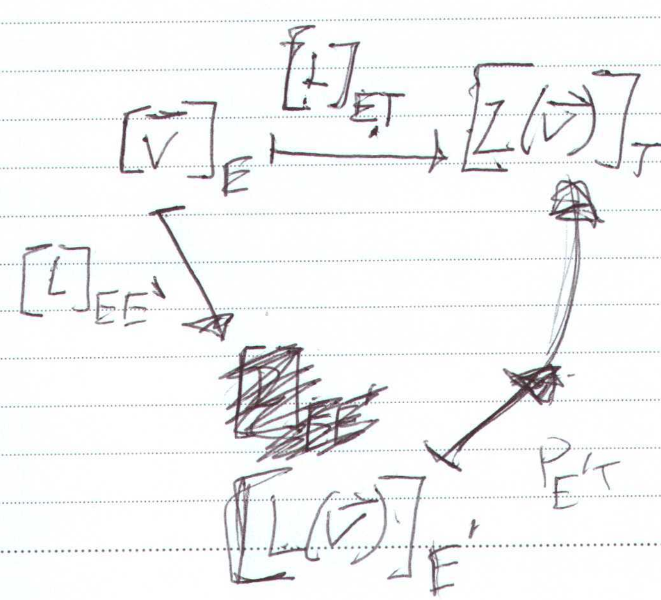
Alternatively

$E' = \{t, 1\}$ for P_1 $\qquad [L]_{EE'} = \begin{bmatrix}-1 & 1\\2 & 1\end{bmatrix}$

$$[L]_{ET} = P_{E'T}[L]_{EE'}$$

$$P_{E'T}\Big(\underbrace{[L]_{EE'}[\vec{v}]_B}_{[L]_{E'}}\Big)$$

$$\begin{bmatrix}1 & 2\\1 & 1\end{bmatrix}^{-1}\begin{bmatrix}-1 & 1\\2 & 1\end{bmatrix}$$



Exercise

Let $L: \mathbb{R}^3 \to \mathbb{R}^3$ be the lin

$$L\left(\underbrace{\begin{bmatrix}1\\1\\0\end{bmatrix}}_{\vec{v}_1}\right) = \underbrace{\begin{bmatrix}0\\1\\1\end{bmatrix}}_{\vec{w}_1}, \quad L\left(\underbrace{\begin{bmatrix}1\\2\\1\end{bmatrix}}_{\vec{v}_2}\right) = \underbrace{\begin{bmatrix}2\\1\\2\end{bmatrix}}_{\vec{w}_2}, \quad L\left(\underbrace{\begin{bmatrix}0\\1\\2\end{bmatrix}}_{\vec{v}_3}\right) = \underbrace{\begin{bmatrix}2\\0\\0\end{bmatrix}}_{\vec{w}_3}$$

Find $L\left(\begin{bmatrix}3\\2\\5\end{bmatrix}\right)$

$L(t^2) = \; 0 \;|\; 2 \;|\; 2$

$L(t) = \; 1 \;|\; 1 \;|\; 0$

$L(1) = \; 1 \;|\; 2 \;|\; 0$

$$\begin{array}{c} t^3 \\ t^2 \\ t \\ 1 \end{array} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 2 \\ 1 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix}$$

$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$ Then compute $\sum_{i=1}^{3} c_i \vec{w}_i$

$\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = [\vec{v}]_S$ $\quad S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

$$[L]_{SE} [\vec{v}]_S = [L]_{SE} \cdot P_{E \to S} [\vec{v}]_E$$

11 29 21

Defn: Let A, B be two square matrices
A is said to be similar to B if there exists an invertible matrix P s.t. $A = PBP^{-1}$

Rmk: In the def, can use $A = P^{-1}BP$ instead
Switch the roles of P and P^{-1} easily

If A is similar to B, then B is similar to A
$A = PBP^{-1} \iff B = P^{-1}AP$

Proposition: If A & B are similar then $\det(A) = \det(B)$

pf: $\det(A) = \cancel{\det(P)} \det(B) \cancel{\det(P)^{-1}}$
$= \det(B)$

Let $L: V \to V$ (domain = codomain)
Let S, S' be two bases for V

Then $[L]_S$ is similar to $[L]_{S'}$

pf:
$[L]_{S'} = P_{S \to S'} [L]_S \underbrace{P_{S' \to S}}_{P} = P^{-1} [L]_S P$

Two matrices are similar iff they represent the same lin. transformation from some V to itself w.r.t different basis

Recall in general if $A = PBQ$ for P, Q invertible then A, B have the same rank, nullity and invertibility. This is particularly applies to similar matrices

TIPS Suppose $S = \{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis for $\mathbb{R}^n$ then $P_{S \to E} = [\vec{v}_1 \cdots \vec{v}_n]$

$$L : \underset{S}{V} \to \underset{T}{W}$$

$$[L]_{S,T} [\vec{v}]_S = [L(\vec{v})]_T \quad \text{for} \quad [L]_{S,T} = \left[[L(\vec{v}_1)]_T \cdots [L(\vec{v}_n)]_T \right]$$

$$\begin{tikzcd}
{[\vec{v}]_{S'}} \arrow[r, mapsto, "{[L]_{S'T'}}"] \arrow[d, "P_{S' \to S}"'] & {[L(\vec{v})]_{T'}} \arrow[d, "P_{T' \to T}"] \\
{[\vec{v}]_S} \arrow[r, mapsto] & {[L(\vec{v})]_T}
\end{tikzcd}$$

Example

$$P = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$S = \left\{ \vec{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right\}$$

$$P \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

S-Coord of $\begin{bmatrix} 3 \\ 7 \end{bmatrix}$

Standard E-Coord $\begin{bmatrix} 3 \\ 7 \end{bmatrix}$ in $\mathbb{R}^2$

A computational trick

Recall if A is invertible

$$RREF([A \mid I_n]) = [I_n \mid A^{-1}]$$

$$RREF([A \mid B]) = [I_n \mid A^{-1}B]$$

☆ Change of Basis formula

Example $L: \mathbb{R}^n \to \mathbb{R}^m$

$\mathcal{E}_n$ $\mathcal{E}_m$

S T

L is rep. in standard bases by A

we want to find $[L]_{S,T}$

Suppose $S = \{\vec{v}_1, \dots, \vec{v}_n\}$ $T = \{\vec{w}_1, \dots, \vec{w}_m\}$

☆ $[L]_{S,T} = P_{\mathcal{E}_m \to T} [L]_{\mathcal{E}_n \mathcal{E}_m} P_{S \to \mathcal{E}_n}$

$= P_{\mathcal{E}_m \to T} [L]_{S, \mathcal{E}_m}$

$[L]_{S, \mathcal{E}_m} = [L(\vec{v}_1) \dots L(\vec{v}_n)]$

$P_{\mathcal{E}_m \to T} = [\vec{w}_1 \dots \vec{w}_m]^{-1}$

b/c

$P_{T \to \mathcal{E}_m} = [\vec{w}_1 \dots \vec{w}_m]$

$\therefore [L]_{ST} = [\vec{w}_1 \dots \vec{w}_m]^{-1} [L(\vec{v}_1) \dots L(\vec{v}_n)]$

$RREF([\vec{w}_1 \dots \vec{w}_m \mid L(\vec{v}_1) \dots L(\vec{v}_n)])$

$= [I_m \mid [L]_{ST}]$

Suppose $L: \mathbb{R}^2 \to \mathbb{R}^2$ is given by $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$S = \{ \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \}$

$L(\vec{v}_1) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$

$[L]_S$

$L(\vec{v}_2) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$

find

$RREF\left(\left[\begin{array}{cc|cc} 1 & 1 & 3 & 5 \\ 1 & 2 & 7 & 11 \end{array}\right] \right) \to \left[\begin{array}{cc|cc} 1 & 1 & 3 & 5 \\ 0 & 1 & 4 & 6 \end{array}\right] \to \boxed{\left[\begin{array}{cc|cc} 1 & 0 & -1 & -1 \\ 0 & 1 & 4 & 6 \end{array}\right]}$

Ans

Reverse $[L]_S = \begin{bmatrix} -1 & -1 \\ 4 & 6 \end{bmatrix}$

$[L]_E$

$= P_{S \to E} [L]_S P_{E \to S}$

$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1}$

Proof

~~$(A^{-1}B)^T = B^T (A^{-1})^T$~~

$B^T (A^T)^{-1}$

$R^6 \rightarrow R^9$: RREF (6×9) matrix

$L: M_{23} \rightarrow M_{33}$

6.2 #7

$$L(A) = \begin{bmatrix} 2 & -1 \\ 1 & 2 \\ 3 & 1 \end{bmatrix} A$$

dim Ker / dim range

$L(A) = 0$ solve for kernel

$L: P_2 \rightarrow R_2$

Kernel?

$L(at^2 + bt + c) = [a\ b] = [0\ 0]$

$$(0)t^2 + (0)t + c = \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix}$$

Kernel Span$\left\{\begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix}\right\}$

range

Rank : dim(range) + dim(Ker)

$3 = \text{dim(range)} + 1$

range of R_2

dim(range) $= 2$

span$\{[1\ 0], [0\ 1]\}$

#6.5 #9

$L: R_3 \to R_2$

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 1 & 0 \end{bmatrix}$$

$S = \{[1\ 0\ -1], [0\ 2\ 0], [1\ 2\ 3]\}$

$T = \{[1\ -1], [2\ 0]\}$

find representation of L wrt the std. bases

$\mathcal{E}_2, \mathcal{E}_3$

$[1\ -1] = [1\ 0] - [0\ 1]$

$[2\ 0] = 2[1\ 0] \quad \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$

$\|$

$B = P_{T \to \mathcal{E}_2} \circ A \circ P_{\mathcal{E}_3 \to S}$

$\|$

Recall

$$\begin{tikzcd} R_3(S) \arrow[r, "A"] & R_2(T) \arrow[d, "P_{T \to \mathcal{E}_2}"] \\ \mathcal{E}_3 \cdot R_3 \arrow[u, "P_{\mathcal{E}_3 \to S}"] \arrow[r] & R_2(\mathcal{E}_2) \end{tikzcd}$$

$[1\ 0\ 0] = x[1\ 0\ -1] + y[0\ 2\ 0] + z[1\ 2\ 3]$

$$\begin{cases} x + z \\ 2y + 2z = 0 \\ -x + 3z = 0 \end{cases} \quad \begin{cases} x = \frac{3}{4} \\ y = -\frac{1}{4} \\ z = \frac{1}{4} \end{cases}$$

$$\begin{bmatrix} 3/4 \\ -1/4 \\ 1/4 \end{bmatrix}$$

Eigen Analysis

Motivation: Find high powers of linear matrices very efficiently

e.g. $A_{2\times 2}$ $\quad R^2 \to R^2$ $\quad A^{1000}$

$A = R_\theta \qquad A^{1000} = R_{1000\theta}$

Example: Suppose $A = \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix}$

find A^{100}

"diagonal matrix"

let $D = \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix} \xrightarrow[\text{implies}]{} D^2 = \begin{bmatrix} 25 & 0 \\ 0 & 16 \end{bmatrix} \ldots D^3 = \begin{bmatrix} 125 & 0 \\ 0 & 64 \end{bmatrix}$

$\ldots D^n = \begin{bmatrix} 5^n & 0 \\ 0 & 4^n \end{bmatrix}$

Justifies

$A = P^{-1}DP \qquad P = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

(A is similar to D)

$A^2 = (P^{-1}D\cancel{P})(\cancel{P^{-1}}DP) = P^{-1}D^2P$

$A^3 = (P^{-1}D^2\cancel{P})(\cancel{P^{-1}}DP) = P^{-1}D^3P$

~~Fact Must Square ma~~

Definition: Let A be a square matrix

A <u>diagonalization</u> of A is an expression of the form $A = P^{-1}DP$ s.t. D is diagonal and P is invertible

Not all matrices are diagonalizable ... most are

A is said to be <u>diagonalizable</u> if it admits a diagonalization

Question:

$$\begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}^{-1}$$

$$\underset{P}{\shortparallel} \qquad \underset{D}{\shortparallel} \qquad \underset{P^{-1}}{\shortparallel}$$

$$A\vec{v_i} = PD\boxed{P^{-1}\vec{v_i}}$$

$$= PD\boxed{\vec{e_i}}$$

$$= P\lambda_i\vec{e_i}$$

$$= \lambda_i P\vec{e_i} = \lambda_i\vec{v_i} \quad \text{for } \exists\, \vec{v}_i$$

Def^n Let A be a square matrix. An eigenvalue for A is a number λ s.t. $\exists\ \vec{v}\ \ A\vec{v} = \lambda\vec{v}$

- Subspace

$$\{\vec{v} \mid A\vec{v} = \lambda\vec{v}\} = \{\vec{v} \mid (A - \lambda I_n)\vec{v} = 0\}$$

$\text{eigen}(\lambda)$ $\qquad$ $\text{Ker}(A - \lambda I_n)$

If for some $\vec{v} \neq \vec{0}$, we have $A\vec{v} = \lambda\vec{v}$

and hence $A - \lambda I_n$ is not invertible

TÜL

Example $A = \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix}$

$$P_A(\lambda) = \det(A - \lambda I_2)$$

$$A - \lambda I_2 = \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6-\lambda & -1 \\ 2 & 3-\lambda \end{bmatrix}$$

$$= (6-\lambda)(3-\lambda) + 2$$

$$= \lambda^2 - 9\lambda + 18 + 2$$

$$= \lambda^2 - 9\lambda + 20 = (\lambda - 5)(\lambda - 4)$$

so $P_A(\lambda) = 0$ when $\lambda = 5 \,\&\, 4$ – eigen values

$$\text{eigen}(5) = \text{Ker}(A - 5I_2) = \text{Ker}\left(\begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}\right) = \text{Span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$

$$\text{eigen}(4) = \text{Ker}(A - 4I_2) = \text{Ker}\left(\begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}\right) = \text{Span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$$

$\lambda_1 = 5 \quad \lambda = 4$

$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1}$

If $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\ker\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} -b \\ a \end{bmatrix}\right)$

if nonzero

Let A be a square matrix $n \times n$

$\boxed{A\vec{v} = \lambda\vec{v}}$ λ is scaler characteristic poly.

$p_A(\lambda) = 0$ ☆ $\det(A - \lambda I)$

$\rightarrow A\vec{v} - \boxed{\lambda\vec{v}} = 0$

$\rightarrow \lambda\vec{v} = \boxed{\lambda I_n \vec{v}}$

$(A - \lambda I_n)\vec{v} = 0$

Suppose

$P = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix} \quad D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

$A\vec{v}_i = PDP^{-1}\vec{v}_i = PD\vec{e}_i = P(\lambda\vec{e}_i) = \lambda P\vec{e}_i = \lambda_i \vec{v}_i$

geometric multiplicity

Algebraic Multiplicity

A is diagonalizable iff it has a diagonal basis (i.e. a basis $S = \{\vec{v}_1, \dots, \vec{v}_n\}$ for $\mathbb{R}^n$ s.t. each $\vec{v}_i$ is an eigenvector for A)

Defn Let λ be an eigenvalue for A.

The algebraic multiplicity of λ is the highest power of $(x-\lambda)$ in $P_A(x)$

The geometric multiplicity of λ is dim eigen(λ)

example

$P_A(x) = (1-\lambda)^2$ $am = (2)$ $gm(\lambda) = 1$

Fact $am(\lambda) \geq gm(\lambda) \geq 1$

Theorem: A is diagonalizable $a.m.(\lambda) = gm(\lambda)$ for all eigen

---

Trivial Example: A is diagonalizable w/ an eigenvalue multiplicity > 1

$\text{eigen}(0) = \ker\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right) = \mathbb{R}^2$

any two lin. dep. $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^2$ from an eigen basis

12 6 21

(1) Complex eigen values/vectors
(2) Inner products

Recap $\mathbb{C}$ = set of complex numbers

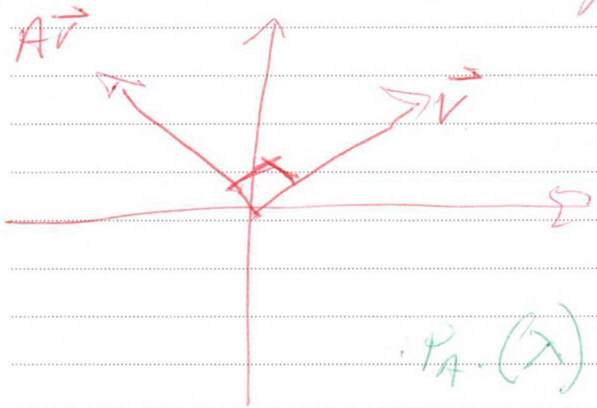
$z \in \mathbb{C}$ is of the form $a+bi$ $a,b \in \mathbb{R}$

Ex $(1+2i)+(2+3i) = 3+5i$

Ex $(1+2i)(2+3i) = 2+2i \cdot 2 + 1 \cdot 3i + (2i)(3i)$
$= 2+7i-6 = -4+7i$

Example

Diagonalize $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$



$A\vec{v} = \lambda \vec{v}$

$\vec{v}$ is a vector such that $\vec{v}$ and $A\vec{v}$ are co-linear

$p_A(\lambda) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1$

eigenvalues $= \pm i$

eigen(i) $= \ker \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} = \mathrm{Span}\left(\begin{bmatrix} -1 \\ i \end{bmatrix}\right)$ ~~flip~~ and negate one value

eigen$(-i)$ $= \ker \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} = \mathrm{Span}\left(\begin{bmatrix} 1 \\ i \end{bmatrix}\right)$

$A = \begin{bmatrix} -i & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} i & i \\ i & 1 \end{bmatrix}^{-1}$

You can still compute A^n quickly.

12 6 21

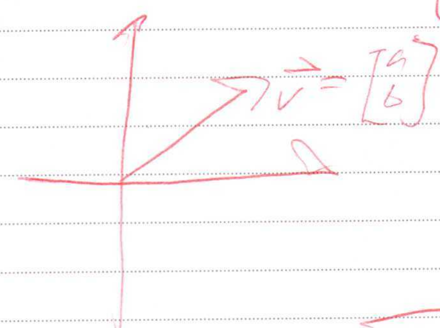
Rmk: #1 $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is not diagonizable even over $\mathbb{C}$

#2 $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is not diagonizable over $\mathbb{R}$ but is diagonizable over $\mathbb{C}$

Unless specified otherwise "diagonizable" means diagonizable over $\mathbb{C}$ by default

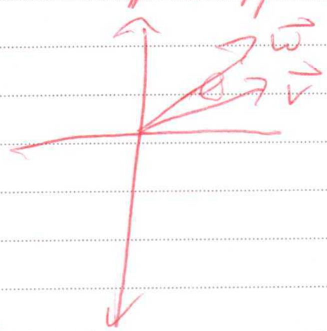
Inner Products

Review dot Products



(norm) length of $\vec v = \sqrt{a^2+b^2} = \sqrt{\vec v \cdot \vec v}$

$\| \vec v \|$



$\cos\theta = \dfrac{\vec v \cdot \vec w}{\|\vec v\| \, \|\vec w\|}$

$\vec v \cdot \vec w = \|\vec v\| \cdot \|\vec w\| \cdot \cos\theta$

Giving dot product $\Longleftrightarrow$ giving length + angles

The same formula work for $\mathbb{R}^n$ (any dimension)

$\underbrace{\mathbb{R}^n}_{\text{generalizing vector space}}$

generalizes dot product

Def^n: For a vector space V, an inner product is a rule which associates two vectors $\vec v, \vec w$ a number $\langle \vec v, \vec w \rangle \in \mathbb{R}$, which behaves like the dot product for $\mathbb{R}^n$

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(a) $\langle \vec{v}, \vec{v} \rangle \geq 0 \quad \langle \vec{v}, \vec{v} \rangle = 0 \iff \vec{v} = 0$

$\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$ If $\|\vec{v}\| = 0 \quad \vec{v} = 0$

(b) Symmetric $\langle \vec{v}, \vec{w} \rangle = \langle \vec{w}, \vec{v} \rangle$

(c) linearity

linear comb. with one entry

$\langle c_1 \vec{v}_1 + c_2 \vec{v}_2, \vec{w} \rangle = c_1 \langle \vec{v}_1, \vec{w} \rangle + c_2 \langle \vec{v}_2, \vec{w} \rangle$

$\langle \vec{v}_1, c_1 \vec{w}_1 + c_2 \vec{w}_2 \rangle = c_1 \langle \vec{v}_1, \vec{w}_1 \rangle + c_2 \langle \vec{v}_1, \vec{w}_2 \rangle$

Example $C^\infty([0,1])$ = space of smooth functions $[0,1] \to \mathbb{R}$

then $\langle f, g \rangle = \int_0^1 f(x) g(x)\, dx$ defines an inner product

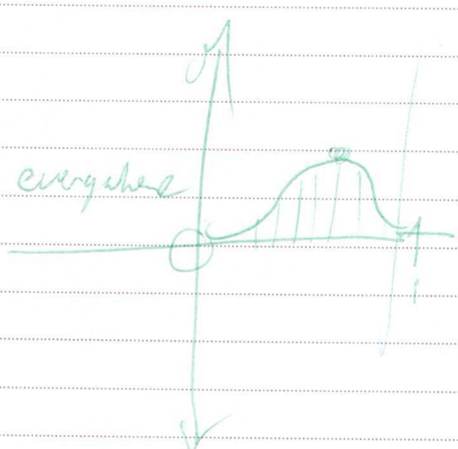
(a) $\langle f, f \rangle = \int_0^1 f(x)^2\, dx$

$f(x)^2 \geq 0 \Rightarrow \int_0^1 f(x)^2\, dx \geq 0$

Suppose that $\int_0^1 f(x)\, dx = 0$

Claim $f(x)^2 = 0 \iff f(x) = 0$ everywhere

$f = 0$



(b) $\langle f, g \rangle = \langle g, f \rangle$

$\int_0^1 f(x)\, dx = \int_0^1 g(x) f(x)\, dx$

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In an inner product space length and angles are defined by the usual formula ......

Consider $C([0,1])$ 2 $\langle\ ,\ \rangle$

$f = x^2$ $\quad \|f\| = \sqrt{\langle f, f\rangle}$ ← length $= \sqrt{\int_0^1 x^4\,dx} = \sqrt{\frac{x^5}{5}\Big|_0^1} = \sqrt{\frac{5(1)-5(0)}{5}}$

$= \frac{1}{\sqrt{5}}$

Rmk: Eigenanalysis works for a general lin trans $L: V \to V$ $\quad L(\vec{v}) = \lambda \cdot \vec{v}$ 12/8/21

eigenvalue eigenvector

example $C^\infty(\mathbb{R})$ = space of smooth functions $\mathbb{R} \to \mathbb{R}$

$L: C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$

$L(e^x) = (e^x)' = e^x$

$\Rightarrow f = e^x$ is an eigenvector w/ eigenvalues

$f = 5 \ldots\ldots\ldots 0$

$L(5) = 0$

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We normally don't think of $\vec{0}$ as an eigenvector, though $\vec{0}$ does lay in every eigenspace

$0 \in \mathbb{R}$ is a perfectly good eigenvalue $eigen(0) = \ker(A)$
$eigen(\lambda) = \ker(L - \lambda I)$

$I(\vec{v}) = \vec{v}$

Def^n. Let V be a finite dim VS.
Let S be a basis for V
Then the characteristic poly of L is defined to be char. of $[L]_S$

$$[L]_S = P_{T \to S} [L]_T P_{S \to T} \Rightarrow [L]_S \wedge [L]_T \text{ are similar}$$

Proposition: Similar matrices have char poly (and have same eigenvalues)

Proof: $A = PBP^{-1}$

$$\begin{aligned}
P_A(\lambda) &= \det(A - \lambda I) \\
&= \det(PBP^{-1} - P(\lambda I_n)P^{-1}) \\
&= \det(P(B - \lambda I_n)P^{-1}) \\
&= \det(P)\det(B - \lambda I_n)\det(P^{-1}) \\
&= \det(B - \lambda I_n) \\
&= P_B(\lambda)
\end{aligned}$$

Caution: A & B don't have the same eigenspace in general
If $A = PBP^{-1}$ $eigen_A(\lambda)$ is the image of $eigen_B(\lambda)$ under P

$$\vec{v} \in eigen_B(\lambda) \Leftrightarrow P\vec{v} \in eigen_A(\lambda)$$

$$\Updownarrow$$

$$B\vec{v} = \lambda\vec{v} \longrightarrow A(P\vec{v}) = \lambda P\vec{v}$$

$$\begin{aligned}
A(P\vec{v}) &= PBP^{-1}(P\vec{v}) \\
&= PB\vec{v} = P(\lambda\vec{v}) = \lambda(P\vec{v})
\end{aligned}$$

The dot product is also called the standard inner product for $\mathbb{R}^n$

$\vec{x} \cdot \vec{y} = \vec{x}^T \vec{y}$ $\quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_3 \end{bmatrix}$

$= \sum_{i=1}^{n} x_i y_i$

Example (non-standard inner product of $\mathbb{R}^2$)

$M = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ Def. $\langle \vec{x}, \vec{y} \rangle = \vec{x}^T M \vec{y}$

(a) $\langle \vec{x}, \vec{x} \rangle \geq 0$ $\quad = 0$ only when $\vec{x} = 0$

$\left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\rangle = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2y_1 + y_2 \\ y_1 + 2y_2 \end{bmatrix}$

M symmetric $\iff M = M^T$

$= x_1(2y_1 + y_2) + x_2(y_1 + 2y_2)$

continued

$\left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle = x_1(2x_1 + x_2) + x_2(x_1 + 2x_2)$

$= 2x_1^2 + 2x_1x_2 + 2x_2^2$

$= x_1^2 + x_2^2 + (x_1 + x_2)^2 \geq 0$ $= 0$ only when $x_1 = x_2 = 0$, $\vec{x} = 0$

(b) want: $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle = \vec{y}^T M \vec{x}$

$\vec{x}^T M \vec{y} = (\vec{x}^T M \vec{y})^T = \vec{y}^T M^T \vec{x}$

(c) $\langle a\vec{x} + b\vec{y}, \vec{z} \rangle = a\langle \vec{x}, \vec{z} \rangle + b\langle \vec{y}, \vec{z} \rangle$

$(a\vec{x} + b\vec{y})^T M \vec{z} = a\vec{x}^T M \vec{z} + b\vec{y}^T M \vec{z}$

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Theorem: If, M is a $n \times n$ matrix which is ★ symmetric ($M = M^T$)

★ positive definite ($\vec{x}^T M \vec{x} \geq 0$; only when $\vec{x} = 0$)

then $\langle \vec{x}, \vec{y} \rangle = \vec{x}^T M \vec{y}$ is an inner product

Conversely all inner products on $\mathbb{R}^n$ come that way

How to recover M from $\langle , \rangle$?

$\langle \vec{x}, \vec{y} \rangle = \vec{x}^T M \vec{y}$

$\langle \vec{e}_i, \vec{e}_j \rangle = \vec{e}_i^T (M \vec{e}_j) = \vec{e}_i^T (j^{th}$ column of $M)$

$= M_{ij}$ $(i,j)^{th}$ entry of M)

Theorem (Spectral Thm)

Let A be an $n \times n$ matrix, symmetric

(a) A is always diagonalizable over $\mathbb{R}$

(all eigenvalues and eigenvectors are real)

(b) A is positive definite iff $\forall$ eigenvalues > 0

example

$P_M(\lambda) = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 3$

$= (\lambda - 3)(\lambda - 1) =$

$\lambda = 1, 3$

★ Eigenspaces are orthogonal to each other

Proof

$\mathbb{R}^n$ has an eigenbasis wrt A

$\vec{v}_1 \ldots \vec{v}_n$ $\quad A\vec{v}_i = \lambda_i \vec{v}_i \quad \lambda_i > 0$

$\vec{v} \in \mathbb{R}^n \quad \sum_{i=1}^{n} c_i \vec{v}_i$

$$\vec{v}^T A \vec{v} = \left(\sum_{i=1}^{n} c_i \vec{v}_i\right)^T \cdot A \cdot \left(\sum_{i=1} c_i \vec{v}_i\right)$$

$$= \sum c_i^2 \vec{v}_i^T \underbrace{A\vec{v}_i}_{\lambda_i \vec{v}_i} = \sum c_i^2 \lambda_i$$

Spectral theorem

Let M be a real symmetric matrix

(a) M is diagonalizable over $\mathbb{R}$

(b) For two different eigenvalues λ, λ'

eigen(λ) $\perp$ eigen(λ') wrt std. inner basis

(c) M is pos. def $\Longleftrightarrow$ all its eigenvalues > 0

M pos def ~~matrix~~ means $\vec{x}^T M \vec{x} \geq 0$

inner product

$0 =$ only when $\vec{x} = 0$

Last time $\langle \vec{x}, \vec{y} \rangle = \vec{x}^T M \vec{y}$

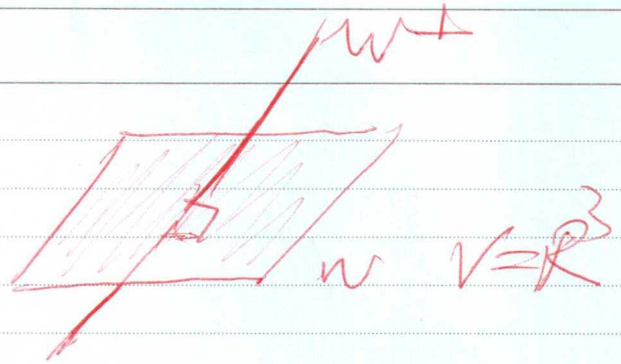
$\Longleftrightarrow$ M is symmetric and pos def.

Orthogonal projection

Let V be an inner product space

$W^\perp = \{\vec{v} \in V \mid \langle \vec{v}, \vec{w} \rangle = 0, \forall \vec{w} \in W\}$

$\cancel{\text{\#}}$ also a subspace

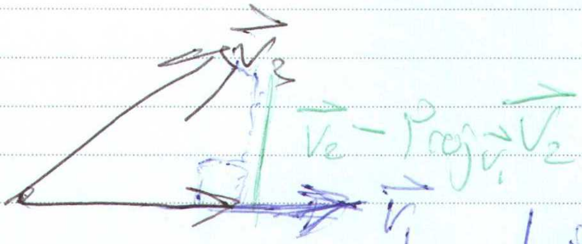


Prop^n $W \cap W^\perp = \{0\}$

Pf. Suppose $\vec{v} \in W \cap W^\perp$

By defin $\langle \vec{w}, \vec{v} \rangle = 0$

$\Rightarrow \vec{v} = 0$



$proj_{\vec{v}_1}\vec{v}_2$

Prop^n There exists a unique vector denoted by $proj_{\vec{v}_1}\vec{v}_2$ w/ the property that $proj_{\vec{v}_1}\vec{v}_2 \in Span(\vec{v}_1)$ and $\langle \vec{v}_2 - proj_{\vec{v}_1}\vec{v}_2, \vec{v}_1 \rangle = 0$

- A unit vector is a vector with length 1.
- The unit vector of $\vec{v}$ is $\frac{\vec{v}}{\|\vec{v}_1\|}$

$proj_{\vec{v}_1}\vec{v}_2 = \|proj_{\vec{v}_1}\vec{v}_2\| \cdot \frac{\vec{v}_1}{\|\vec{v}_1\|}$

$= \|\vec{v}_2\| \cos\theta \frac{\vec{v}_1}{\|\vec{v}_1\|} = \|\vec{v}_2\| \cancel{\frac{\vec{v}_1 \cdot \vec{v}_2}{\|\vec{v}_1\|\|\vec{v}_2\|} \frac{\vec{v}_1}{\|\vec{v}_1\|}}$

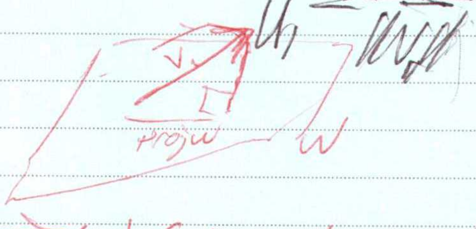
$= \frac{\vec{v}_1 \cdot \vec{v}_2}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1$

In general inner product space

$proj_{\vec{v}_1}\vec{v}_2 = \frac{\langle \vec{v}_1, \vec{v}_2 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle} \cdot \vec{v}_1$

$= \langle \vec{v}_2, \vec{u}_1 \rangle \cdot \vec{u}_1$

$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$



Similarly $W \subseteq V$ subspace $\vec{v} \in V$ we talk about the unique vector in W s.t. $\vec{v} - proj_{\vec{v}} W \in W^\perp$ is the shortest distance between $\vec{v}$ and points in W.

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Definition A Basis $S = \{\vec{u}_1 \dots \vec{u}_n\}$ is an orthonormal basis iff $\|\vec{u}_i\| = 1$, $\langle \vec{u}_i, \vec{u}_j \rangle = 0 \; \forall i \neq j$

Example the standard basis wrt the inner product

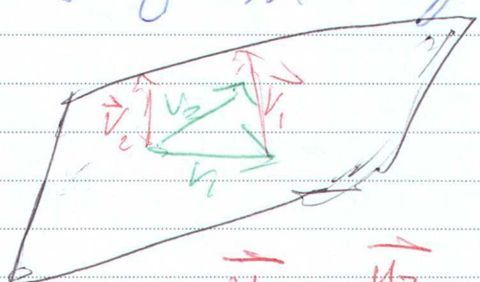
prop. any fin-dim vector space has an orthonormal basis.

If $\{\vec{u}_1 \dots \vec{u}_m\}$ is an orthonormal basis for W then $proj_W \vec{v} = \sum_{i=1}^{m} \langle \vec{v}, \vec{u}_i \rangle \vec{u}_i$

does not hold if the basis is not orthonormal

Gram-Schmidt

An algorithm which produces an orthonormal basis from any basis from any basis



$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$

$\vec{u}_2' = \vec{v}_2 - \langle \vec{v}_2, \vec{u}_1 \rangle \vec{u}_1$

$\vec{u}_2 = \frac{\vec{u}_2'}{\|\vec{u}_2'\|}$

$\vec{u}_3 = \vec{w}_3 - \langle v_3, u_1 \rangle \vec{u}_1 - \langle \vec{v}_3, \vec{u}_2 \rangle \vec{u}_2$

$\vec{u}_3 = \frac{\vec{u}_3}{\|\vec{u}_3\|}$

$\vec{u}_4 = \vec{v}_4 - \sum_{i=1}^{3} \langle \vec{v}_4, u_i \rangle u_i$

$\vec{u}_4 = \frac{\vec{u}_4}{\|\vec{u}_4\|}$ .....

Don't forget the normalization

Prop^n V be an inner ① product space

$W \subseteq V$ $W^\perp$

ⓐ $\dim W + \dim W^\perp = \dim V$

ⓑ If S is a bases for W and S' is a bases for $W^\perp$ then.

~~$S \cup$~~ $S \cup S'$ is a bases for V

Proof ⓐ Assume that $S = \{\vec{u}_1, \vec{u}_2, \ldots \vec{u}_m\}$ define a linear transformation $L: V \longrightarrow \mathbb{R}^m$ by. $L(\vec{v}) = \begin{bmatrix} \langle \vec{v}, \vec{u}_1 \rangle \\ \langle \vec{v}, \vec{u}_m \rangle \end{bmatrix} \in \mathbb{R}^m$

$m = \dim W$ is orthonormal

claim L is onto

$\operatorname{im}(L) \supseteq \operatorname{span}(L(\vec{u}_1) \ldots L(\vec{u}_m))$

$L(\vec{u}_1) = \vec{e}_1 \in \mathbb{R}^m$ $\mathbb{R}^m$

$\ker(L) = W^\perp$

$\underbrace{\dim \ker(L)}_{\dim W^\perp} + \underbrace{\dim \operatorname{im}(L)}_{\dim W} = \dim V$

by rank-nullity